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Algebraic Derivation of "Broken $\mathbb{Z}_N$- Symmetric" Solutions for the Star-Triangle Relations

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1. Introduction. Rescently remarkable progress is made in the theory of exactly solvable lattice models. Among them well-studied are the models defined by the quantum R- matrices which is the solution of the Yang - Baxter equation (YBE) related to the q-analogue of the enveloping algebra $U_q(\mathfrak{g})$ [J]. Meanwhile there are solutions for the YBE or the STR whose algebraic background were not clarified so far. Fateev -Zamolodchikov's $\mathbb{Z}_N$- symmetric model[FZ], chiral Potts model [AMPTY][BPA] and Kashiwara - Miwa's broken $\mathbb{Z}_N$-symmetric model* [KM] are for example. In these three solutions the latter two are mutually different extention of the first one: while chiral Potts is defined by the coordinate functions of some curve of genus > 1 , broken $\mathbb{Z}_N$ is defined by the elliptic theta functions.

Rescently Bazhanov and Stroganov observed that the chiral Potts can be obtained as a solution of some linear equation related to the 6- vertex model [BS]. In this note, sketching [HY], we will show that the broken $\mathbb{Z}_N$ model can be obtained as a descendant of the 8-vertex model in the same manner. For that purpose we use Sklyanin's L- operator for the 8- vertex model instead of the L- operator which is given and used by [BZ] to get the chiral Potts. In our derivation a new discrete parameter $N' \in \mathbb{Z}$, $(N,N')=1$ (coprime), which is involved in Sklyanin's L naturally come into the broken $\mathbb{Z}_N$ model.

(*: This is an extention of $\mathbb{Z}_N$- symmetric solution of the STR but not $\mathbb{Z}_N$ symmetric, hence the name. ($\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$.) In [KM] they found this solution through computer experiments.)

2. *Sklyanin algebra*. As is similar to the method of [BZ] for chiral Potts, we will derive the broken $\mathbb{Z}_N$- model as the solution $S \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ for the equation

1) $$S^{12} L^{02}(u) L^{01}(v) = L^{01}(v) L^{02}(u) S^{12} \quad \text{on} \quad V_0 \otimes V_1 \otimes V_2,$$

where $V_0 = \mathbb{C}^2$, $V_1 = V_2 = \mathbb{C}^N$, $L^{01}(v) = L(v) \otimes id_{V_2}$ etc. Here $L \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^2)$ is a given L- operator which satisfies

2) $$R^{12}(u-v)L^{02}(u)L^{01}(v) = L^{01}(v)L^{02}(u)\, R^{12}(u-v) \quad \text{on} \quad V_0 \otimes V_1 \otimes V_2,$$

where $V_0 = \mathbb{C}^N$, $V_1 = V_2 = \mathbb{C}^2$ and $R(u) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is the Baxter's 8- vertex solution for YBE [Fig.0]

3) $$R^{12}(u-v)R^{02}(u)R^{01}(v) = R^{01}(v)R^{02}(u)\, R^{12}(u-v) \quad \text{on} \quad \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$$

We will solve 1) for Sklyanin's L- operator, which we recall now [S1,S2]. By means of the Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sqrt{-1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we can write Baxter's solution as $R(u) = \Sigma_{\alpha=0}^{3} w_\alpha(u)\, \sigma_\alpha \otimes \sigma_\alpha$, where

$$(w_0,w_1,w_2,w_3)=(H(u+\eta)/H(\eta),H_1(u+\eta)/H_1(\eta),\Theta_1(u+\eta)/\Theta_1(\eta),\Theta(u+\eta)/\Theta(\eta))$$

with $H(u)= 2q^{1/4}\sin \frac{\pi u}{2I}\, \Pi_{n=1}^{\infty}(1-2q^{2n}\cos \pi u/I +q^{4n})(1-q^{2n})$ $(q=e^{-\pi I'/I})$, $\Theta(u)=-\sqrt{-1}\, e^{\sqrt{-1}\pi u/2I}H(u+\sqrt{-1}I')$, $H_j(u):= H(u+jI)$, $\Theta_j(u):= \Theta(u+jI)$ [B]

Sklyanin's observation. Let w_α are as above and $S_\alpha \in \mathrm{Mat}(\mathbb{C}^N)$. Then $L(u) = \Sigma_{\alpha=0}^{3} w_\alpha(u)\, \sigma_\alpha \otimes S_\alpha$ satisfies 2) if and only if the following relations (***without the spectral parameter*** u) hold:

4) $$[S_0, S_\alpha]_+ = -\sqrt{-1}\, [S_\beta, S_\gamma]_-, \quad [S_0, S_\alpha]_- = \sqrt{-1}\, J_{\beta\gamma}\, [S_\beta, S_\gamma]_+ .$$

Here $(\alpha,\beta,\gamma)=(123),(231),(312)$ (so there are 6 relations) and $J_{\beta\gamma} = -J_{\gamma\beta}$, $(J_{12},J_{23},J_{13})=(H\Theta(\eta)^2/H_1\Theta_1(\eta)^2, HH_1(\eta)^2/\Theta\Theta_1(\eta)^2, H\Theta_1(\eta)^2/\Theta H_1(\eta)^2)$.■

We call the associative algebra defined by 4 generators S_α with defining relations 4) the ***Sklyanin algebra***. A representation of the Sklyanin algebra gives an L- operator by the above. Sklyanin's own representation of "series b" [S2] provides the L- operator that we use here, which has continuous parameters γ,λ,Δ and is given by

$$S_\alpha = S_\alpha(\gamma,\lambda,\Delta) \in \mathrm{End}\, \mathbb{C}^N, \quad \mathbb{C}^N = \oplus_{a=0}^{N-1} \mathbb{C}\, \varphi_a, \quad \varphi_a = \varphi_{a \bmod N}$$

5) $S_\alpha \varphi_a := \Theta(A_a)^{-1}\{ \Delta\, s_\alpha(A_a+\eta-\lambda)\varphi_{a+1} + \Delta^{-1}\varepsilon_\alpha s_\alpha(A_a-\eta+\lambda)\varphi_{a-1} \}$,

where $A_a = 2\gamma+2a\eta$, $\varepsilon_0 = -\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, $s_0(u) = H(\eta)\Theta_1(u)$, $s_1(u) = -\sqrt{-1}\, H_1(\eta)\Theta(u)$, $s_2(u) = \Theta_1(\eta)H(u)$, $s_3(u) = \Theta(\eta)H_1(u)$.

This representation is well defined iff $\eta = 2IN'/N$ for some $N' \in \mathbb{Z}$, and can be considered as an elliptic analog of the "cyclic" representations [DK] of the algebra $U_q(\mathfrak{sl}(2))$ at $q = \exp 2\pi i N'/N$.

3. "Localization" of eq. 1). The matrix elements of the L-operator associated to the representation 5) factor as [Fig.1]:

$$L(u)=(L(u)^{ia}_{jb})_{(jb),(ia)} \begin{pmatrix} i,j = 0,1 \\ a,b = 0,\dots,N-1 \end{pmatrix}, \quad L(u)^{ia}_{jb} = K(u)^{a}_{jb}\, K(u)^{ia}_{b}.$$

According to this, we assume that S is of the form [Fig.2]

6) $$S(u,v)^{a\,b}_{a'b'} = W(u,v)_{ab}\, W(u,v)_{b'a'}\, \overline{W}(u,v)^{a}_{b'}\, \overline{W}(u,v)^{b}_{a'}.$$

(So W's and $\overline{W}$'s [Fig.3] are unknown.) Then by [Fig.6] we find

Proposition *("Localization"). The following 7) and 8) are sufficient for* S *of the form 6) to satisfy the equation 1) :*

7) $$W(u,v)_{dc}\Sigma_{j=0}^{1} K(u)^{a}_{jd}\, K(v)^{jb}_{c} = W(u,v)_{ab}\Sigma_{j=0}^{1} K(v)^{a}_{jd}\, K(u)^{jb}_{c} \quad \text{[Fig.4]},$$

8) $$\Sigma_{b=0}^{N-1} K(u)^{b}_{jc}K(v)^{ib}_{c}\, \overline{W}(u,v)^{a}_{b} = \Sigma_{b=0}^{N-1} \overline{W}(u,v)^{b}_{c}\, K(v)^{b}_{jc}K(u)^{ia}_{b} \quad \text{[Fig.5]}. \blacksquare$$

4. **Theorem**[HY]. **A)** *The equation 7(resp. 8) have a nontrivial solution if and only if the following 9 (resp. 10) is satisfied:*

9) $$\lambda \in \mathbb{Z}\cdot 2I/(2,N), \quad 2\gamma \in \mathbb{Z}\eta + \mathbb{Z}I.$$

10) $$\lambda \in \mathbb{Z}I, \quad 2\gamma \in \mathbb{Z}\eta + \mathbb{Z}I, \quad \Delta^{2N} = 1.$$

Under these conditions, up to W_{00} *the solution is determined by*

$$\frac{W(u,v)_{a+1,b+1}}{W(u,v)_{a,b}} = \frac{\Theta_1(\ 2\gamma+(a+b+1)\eta-\lambda-t\ )}{\Theta_1(\ 2\gamma+(a+b+1)\eta-\lambda+t\ )},$$

$$\frac{W(u,v)_{a+1,b-1}}{W(u,v)_{a,b}} = \frac{H_1((a-b+1)\eta-\lambda-t\ )}{H_1((a-b+1)\eta-\lambda+t\ )} \quad (\ t = (u-v)/2\ ),$$

$$\overline{W}((\eta-\lambda-I)-u)^{b}_{a} = (\ \sqrt{-1}^{\ -\lambda/I}\ \Delta^{N-1})^{b-a}\ \Theta(A_a)\ W(u)_{a,b},$$

where $W((u-v)/2) = W(u,v)$, $\overline{W}((u-v)/2) = \overline{W}(u,v)$.

B) *If* $(N,N')=1$, *then the STR hold : for some scalar function* ρ,

$$\Sigma_{d=0}^{N-1} \overline{W}(u-v)^{d}_{c}\, W(u-w)_{ad}\overline{W}(v-w)^{b}_{d} = \rho W(v-w)_{ac}\overline{W}(u-w)^{b}_{c}\, W(u-v)_{ab} \quad \text{[Fig.7]}. \blacksquare$$

Note that the STR for W, $\bar{W}$ imply the YBE 3) for **S** of 6)[Fig.8].

Theorem A is shown by direct calculation. To show B, braid manipulations like [Figs.9,10] give enough relations to reduce the problem to the Schur's Lemma for the Sklyanin algebra representations.

For certain parameters, W and $\bar{W}$ reduce to the solution of STR in [KM]. As in their case[JMO], we expect that the 1 point function of the model (defined by W, $\bar{W}$) relates to affine Lie algebra characters.

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Fig.0

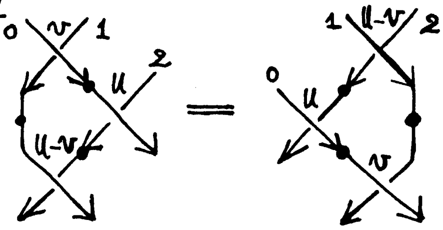
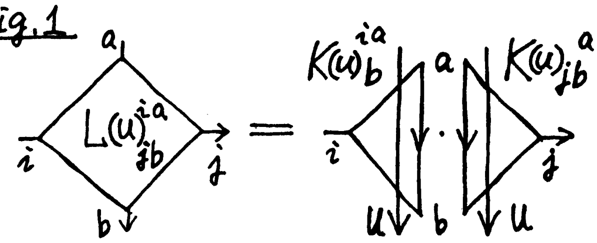


Fig.1



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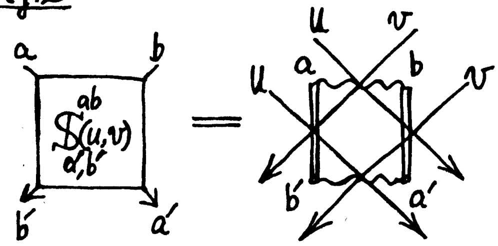

Fig.3

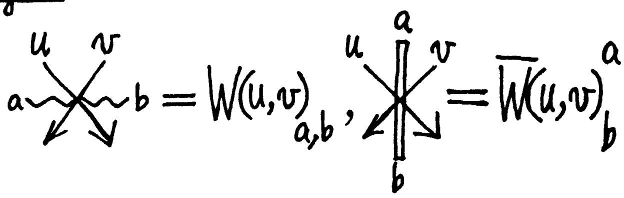

Fig.4

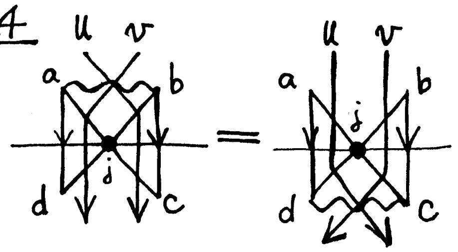

Fig.5

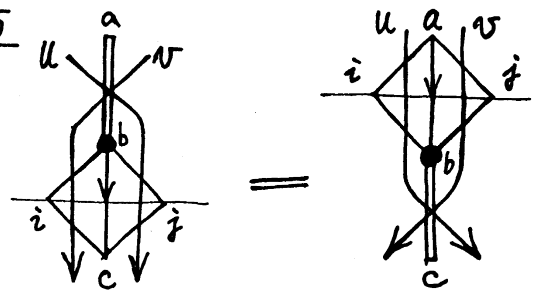

Fig.6

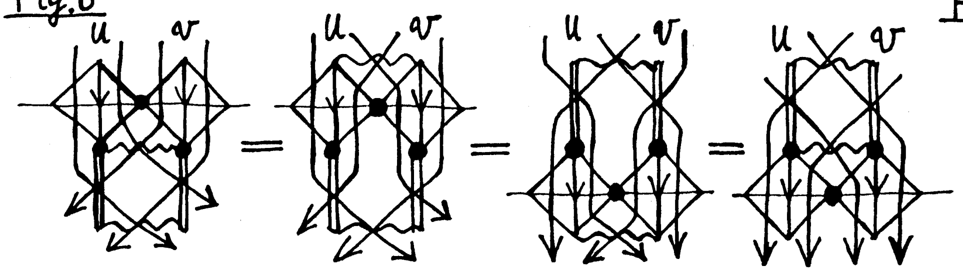

Fig.7

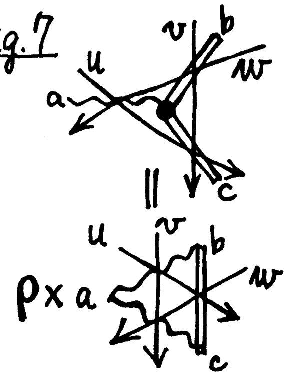

Fig.8

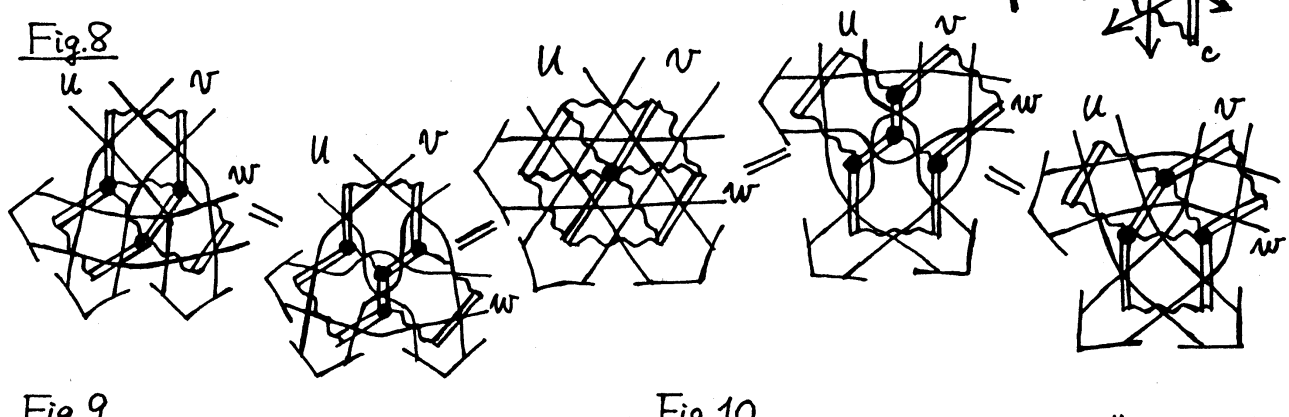

Fig.9

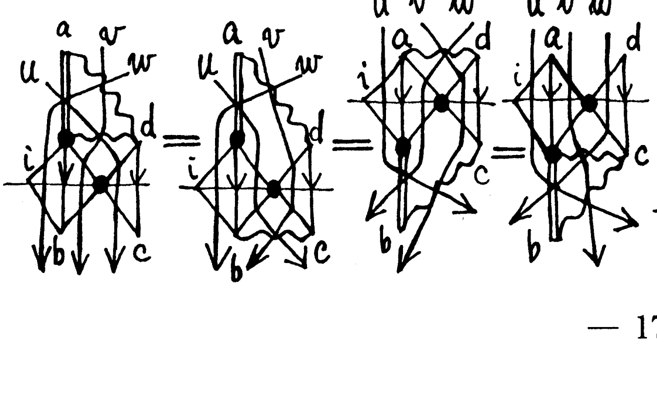

Fig.10