ユニタリ表現論セミナー報告集 X, 1990
pp.10-12

ON SOME CATEGORIES OF $(\mathfrak{g},K)$ - MODULES

A.Guichardet, Ecole Polytechnique,
91128 Palaiseau, France

The present talk is a report on a joint work with P.Y.Gaillard.

§ 1. INTRODUCTION.

The works of Gabriel and Mitchell have established a close connection between some algebras and some categories. More precisely let us call an algebra A (over **C**, with unit) *admissible* if the following conditions are fulfilled, with R being the (Jacobson) radical of A :

- A/R and R/R^2 are finite dimensional
- A is separated and complete for the R-adic topology.

Let us also say that A is *sober* if the algebra A/R is isomorphic to some algebra $\mathbf{C}^m$; there is a procedure associating with each admissible algebra a sober one which we denote by $\mathfrak{X}(A)$.

On the other hand let us call a category $\mathscr{C}$ *admissible* if :

- $\mathscr{C}$ is abelian
- the groups $\mathrm{Hom}_{\mathscr{C}}(E,F)$ are vector spaces
- $\mathscr{C}$ has only a finite number of simple objects E_i , and $\mathrm{End}_{\mathscr{C}}(E_i) = \mathbf{C}$
- every object in $\mathscr{C}$ has finite length
- $\dim \mathrm{Ext}^1_{\mathscr{C}}(E_i,E_j) < +\infty$.

Then the following result is true : by associating to each sober admissible algebra A the category of all finite dimensional left A-modules, one obtains a bijective correspondence between sober admissible algebras and admissible categories.

The aim of this talk is to construct the algebra A in case $\mathscr{C}$ is some category of $(\mathfrak{g},K)$-modules for a semi-simple group G.

§ 2. STATEMENT OF RESULTS.

We denote by

- G a real semi-simple connected Lie group with finite center $Z(G)$
- I a maximal ideal of the algebra $Z(\mathcal{U}(\mathfrak{g})) \otimes \mathbb{C}[Z(G)]$
- $\mathcal{E} = \{E_1,\ldots,E_s\}$ the set of all simple $(\mathfrak{g},K)$-modules annihilated by I
- $\mathcal{C}$ the category of all $(\mathfrak{g},K)$-modules annihilated by some power of I, or else the category of all Harish-Chandra modules having simple subquotients in $\mathcal{E}$; it is admissible.

We choose a finite dimensional K-module X such that

$$n_i = \dim \mathrm{Hom}_K (X,E_i) > 0 \qquad \forall i \quad ;$$

we set

$$\begin{aligned} P &= \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} X && ((\mathfrak{g},K)\text{-module}) \\ A_0 &= \mathrm{End}_{\mathfrak{g},K} (P) && \\ J &= I.A_0 && (\text{two-sided ideal in } A_0) \\ A &= \widehat{(A_0)}_J && (\text{completion of } A_0 \text{ at the ideal } J) . \end{aligned}$$

<u>Theorem</u> 1. (i) *The algebra* A *is admissible and the category* $\mathcal{C}$ *is equivalent to the category of all finite dimensional left* A-*modules*

(ii) $A/\mathrm{rad}\, A$ *is isomorphic to the product* $\prod_{i=1}^{s} M(n_i,\mathbb{C})$

(iii) *the sober admissible algebra of* $\mathcal{C}$ *is* $\mathcal{Z}(A^{op})$, *and it is* A^{op} *itself iff* $n_i = 1\ \forall\, i$

The following result will give some more information about A_0. We denote by

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition
- $\mathcal{U}_n(\mathfrak{g})$ the canonical filtration of $\mathcal{U}(\mathfrak{g})$
- P_n the image of $\mathcal{U}_n(\mathfrak{g})$ in P
- $\mathrm{gr}\, A_0$ the graduated algebra associated with the corresponding filtration of A_0.

<u>Theorem</u> 2. *We have*

$$\begin{aligned} A_0 &= (S(\mathfrak{p}) \otimes \mathrm{End}\, X)^K && (\text{isomorphism of filtrated vector spaces}) \\ \mathrm{gr}\, A_0 &= (S(\mathfrak{p}) \otimes \mathrm{End}\, X)^K && (\text{isomorphism of algebras}) \\ \mathrm{gr}\, A_0 &= S(\mathfrak{p})^K \otimes \mathrm{End}_M X && (\text{isomorphism of vector spaces}) . \end{aligned}$$

§ 3. EXAMPLE.

We take

- $G = SL(2,\mathbf{R})$
- $K = SO(2)$, with characters χ_n, $n \in \mathbf{Z}$
- $\mathcal{E}$ = set of simple $(\mathfrak{g},K)$-modules with trivial infinitesimal and central characters $= \{E_1,E_2,E_3\}$ where E_1 and E_3 are discrete series and E_2 is the trivial $(\mathfrak{g},K)$-module
- $X = C_{\chi_{-2}} \oplus C_{\chi_0} \oplus C_{\chi_2}$ with basis ε_1, ε_2, ε_3; here we have $n_i = 1 \ \forall\, i$
- basis of $\mathfrak{g}_c$:

$$X_\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \in \mathfrak{p}_c \quad , \quad X_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k} .$$

We shall give a description of A_0 in terms of generators (namely of a quiver) and relations. Let us consider the following quiver

$$Q : \qquad e_1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} e_2 \underset{\delta}{\overset{\beta}{\rightleftarrows}} e_3 .$$

Let CQ the algebra of Q and B_0 its quotient by the relation $\alpha\gamma - \delta\beta = 0$; one can define a homomorphism of algebras $\Pi : B_0 \to A_0$ as follows :

$$\Pi(e_i) = 1 \otimes T_{i,i}$$

$$\Pi(\alpha) = X_- \otimes T_{2,1} \quad , \quad \Pi(\beta) = X_- \otimes T_{3,2}$$

$$\Pi(\gamma) = X_+ \otimes T_{1,2} \quad , \quad \Pi(\beta) = X_+ \otimes T_{2,3} ;$$

an easy calculation shows that Π is an *isomorphism* and also that *the sober admissible algebra of* $\mathcal{C}$ *is isomorphic to the completion of* B_0 *at the ideal consisting of all elements of strictly positive degree*.