ユニタリ表現論セミナー報告集 X, 1990
pp.8-9

Plancherel formula for monomial representations of nilpotent Lie groups

Hidenori Fujiwara
Faculty of Technology in Kyushu, Kinki University, Iizuka 820, Japan

Let H be an analytic subgroup of a connected and simply connected nilpotent Lie group G, let χ be a unitary character of H. Concerning the monomial representation $\tau = \mathrm{ind}_H^G \chi$ of G, we consider the Dirac measure δ_τ at the unit element e;

$$\delta_\tau : \mathcal{H}_\tau^{+\infty} \ni \Phi \mapsto \overline{\Phi(e)} \in \mathbf{C},$$

here $\mathcal{H}_\tau^{+\infty}$ is the space of C^∞-vectors for τ. Since $\mathcal{H}_\tau^{+\infty} \subset C^\infty(G)$, δ_τ has a sense and we have, $\mathcal{H}_\tau^{-\infty}$ denoting the antidual space of $\mathcal{H}_\tau^{+\infty}$,

$$\delta_\tau \in (\mathcal{H}_\tau^{-\infty})^{H,\chi} = \{a \in \mathcal{H}_\tau^{-\infty};\ \tau(h)a = \chi(h)a,\ h \in H\}.$$

Now let

$$\tau = \int_{\hat{G}}^{\oplus} m(\pi)\pi d\nu(\pi)$$

be the canonical central decomposition of τ. Accordingly the generalized vector δ_τ of τ is decomposed :

$$\delta_\tau = \int_{\hat{G}}^{\oplus} \sum_{k=1}^{m(\pi)} a_\pi^k \, d\nu(\pi), \qquad (*)$$

with $a_\pi^k \in (\mathcal{H}_\pi^{-\infty})^{H,\chi}$ $(1 \leqq k \leqq m(\pi))$.

Taking a left Haar measure dh on H, we put, for any $\phi \in C_c^\infty(G)$,

$$\phi_H^\chi(g) = \int_H \phi(gh)\chi(h)dh \in \mathcal{H}_\tau^{+\infty}.$$

Then, normalizing a left Haar measure dg on G, which defines

$$\tau(\phi) = \int_G \phi(g)\tau(g)dg,$$

and $\nu_{G,H}$ which gives the norm in $\mathcal{H}_\tau$, the space of τ, in order to satisfy dg/dh $= \nu_{G,H}$, we have

$$\phi_H^\chi(e) = \langle \tau(\phi)\delta_\tau, \delta_\tau \rangle$$

for any $\phi \in C_c^\infty(G)$. Hence, by (*),

$$\phi_H^\chi(e) = \int_{\hat{G}} \sum_{k=1}^{m(\pi)} \langle \pi(\phi)a_\pi^k, a_\pi^k \rangle d\nu(\pi),$$

which we call Penney's Plancherel formula for the monomial representation τ.

On the other hand, there exists a field of nuclear operators $U = \{U(\pi)\}_{\pi\in\hat{G}}$ satisfying

$$\phi_H^\chi(e) = \int_{\hat{G}} \mathrm{tr}(\pi(\phi)U(\pi))d\nu(\pi), \quad \phi \in C_c^\infty(G),$$

which we call Bonnet's Plancherel formula for τ. Clearly these two formulae are related by

$$U(\pi)(v) = \sum_{k=1}^{m(\pi)} \langle v, a_\pi^k \rangle a_\pi^k, \quad v \in \mathcal{H}_\pi^{+\infty}.$$

We try to describe these formulae in the frame of orbit method, and this time we look for $U(\pi)$, putting no assumption on the multiplity $m(\pi)$ in the decomposition of τ into irreducible representations.

References

[1] Y.Benoist, Analyse harmonique sur les espaces symétriques nilpotents, J. Funct. Anal., 59(1984), 211-253.

[2] P.Bonnet, Transformation de Fourier des distributions de type positif sur un groupe de Lie unimodulaire, J. Funct. Anal., 55(1984), 220-246.

[3] H.Fujiwara, Représentations monomiales des groupes de Lie nilpotents, Pacific J. Math., 127(1987), 329-352.

[4] G.Grélaud, La formule de Plancherel pour les espaces homogènes des groupes de Heisenberg, J. für die Reine und Ange. Math..

[5] A.A.Kirillov, Représentations unitaires des groupes de Lie nilpotents, Uspekhi Mat. Nauk, 17(1962), 57-110.

[6] R.Lipsman, The Penney-Fujiwara Plancherel formula for symmetric spaces, Progress in Math..

[7] R.Penney, Abstract Plancherel theorems and a Frobenius reciprocity theorem, J. Funct. Anal., 18(1975), 177-190.