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Connection Formulas for Invariant Eigendistributons on $U(p,q)/\{U(r)\times U(p-r,q)\}$

Shigeru Aoki (Takushoku University)
Suehiro Kato (Kitasato University)

§1. Introduction

Let X be a semisimple symmetric space G/H, and $\mathcal{O}$ a H-invariant open subset of X. Let $\mathbb{D}(X)$ be the ring of invariant differential operators on X, and χ a character of $\mathbb{D}(X)$. A Schwartz distribution Θ on $\mathcal{O}$ is said to be an invariant eigendistribution (I.E.D.) with an infinitesimal character χ, if

(i) Θ is H-invariant, and

(ii) $D\Theta = \chi(D)\Theta$ for all $D\in\mathbb{D}(X)$.

We denote by $\mathcal{D}'_{\chi,H}(\mathcal{O})$ the set of all I.E.D.'s with χ on $\mathcal{O}$.

Let X' be the H-invariant open dense subset of all regular semisimple elements in X. As is well-known, any $\Theta\in\mathcal{D}'_{\chi,H}(X')$ is a real analytic function. For any $\Theta\in\mathcal{D}'_{\chi,H}(X)$, we have $\Theta|_{X'}\in\mathcal{D}'_{\chi,H}(X')$ clearly.

In the following, we put $X = U(p,q)/\{U(r)\times U(p-r,q)\}$. We wish to determine I.E.D.'s on X as explicitly as possible. For this end, we consider the following problem, which originates in the corresponding one for semisimple Lie groups investigated in detail by Hirai [4] :

<u>Problem</u> Find a condition that an I.E.D. on X' is extensible to an I.E.D. on X.

The authors studied this problem in Aoki-Kato [1](based on the results of Faraut [3]) for $r = 1$, and moreover in Aoki-Kato [2] for $p = 4$, $q = r = 2$. In this article, we give a necessary condition in the case of any p, q and r. (For simplicity, we assume some condition ((※) in §2).) It is shown that our necessary condition in this article is also sufficient one, when the infinitesimal charactder χ is generic. We conjecture that this fact holds even in the case where χ is not generic.

Finally, we briefly describe our method. Let x be a singular semisimple element in X. To attach the problem above, we want to know the behavior of I.E.D.'s around x. For this purpose, we need the knowledge of the followings:

① (the radial parts of) invariant differential operators.

② invariant integrals, especially, their behavior around x.

The result of ① is essentially given by Hoogenboom [5] . On the other hand, to get the knowledge of ②, we consider the symmetric subspace $Z_G(x)/Z_H(x)$ produced by the centralizers of x, and the invariant integrals on it.

§2. Preliminaries

Put $U(m,n) = \{g\in GL(m+n,\mathbb{C}) \mid g\, I_{m,n}\, g^* = I_{m,n}\}$ where $I_{m,n} = \mathrm{diag}\,(\overbrace{1,\cdots,1}^{m},\overbrace{-1,\cdots,-1}^{n})$ and $g^* = {}^t\bar{g}$. Let p,q,r be positive integers. We assume, for simplicity, the condition

(※) $2r \leqq p,\ 1 \leqq r \leqq q$.

Let G denote the group $U(p,q)$. For $g \in G$, put $\sigma(g) = I_{r,p+q-r}\, g I_{r,p+q-r}$. Let H denote the subgroup of G consisting of elements fixed by σ - then H is isomorphic to $U(r) \times U(p-r,q)$. Put $X = G/H$. We define an inclusion $\tilde{\sigma}$ of X into G by $\tilde{\sigma}(j) = g\sigma(g)^{-1}$ with $j = gH \in X$. By this map, X is identified with $\tilde{\sigma}(X)$.

Put $J_\ell = \{ \tilde{g j}_{\theta_1,\cdots,\theta_\ell, t_{\ell+1},\cdots,t_r} \mid \theta_1,\cdots,\theta_\ell, t_{\ell+1},\cdots,t_r \in \boldsymbol{R} \}$. $J_0,\cdots,J_r$ form a complete system of representatives of H-conjugate classes in analogues of Cartan subgroups. (J_0 is split and J_r is compact.) So the rank of X, i.e., the dimension of J_ℓ is r. The Weyl group of J_ℓ is isomorphic to $(\mathbb{Z}_2)^r \rtimes (\mathfrak{S}_\ell \times \mathfrak{S}_{r-\ell})$, where we denote by $\mathfrak{S}_n$ the n-th symmetric group. Putting $J_\ell' = J_\ell \cap X'$, we get $X' = \coprod_{\ell=0}^{r} H.J_\ell'$. Let $j = \tilde{g j}_{\theta_1,\cdots,\theta_\ell, t_{\ell+1},\cdots,t_r}$ be an element of J_ℓ -then the τ-parameter of j is the n-tiple of real numbers $(\tau_1,\cdots,\tau_r)$ defined by $\tau_i = \cos^2\theta_i$ $(1 \leqq i \leqq \ell)$, $\tau_i = \cosh^2 t_i$ $(\ell+1 \leqq i \leqq r)$. By the mapping $j \mapsto (\tau_1,\cdots,\tau_r)$, J_ℓ is mapped onto the set

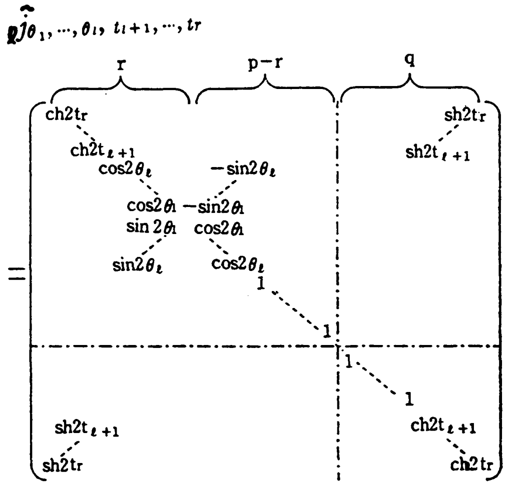

Fig.

$(**)$ $\{(\tau_1,\cdots,\tau_r) \mid 0 \leqq \tau_i \leqq 1\ (1 \leqq i \leqq \ell),\ 1 \leqq \tau_i\ (\ell+1 \leqq i \leqq r)\}$

and $j \in J_\ell'$ iff $\tau_i \neq 0,1$ $(1 \leqq i \leqq r)$, $\tau_i \neq \tau_j$ $(1 \leqq i,j \leqq r;\ i \neq j)$. Corresponding to the action of the Weyl group, the group $\mathfrak{S}_\ell \times \mathfrak{S}_{r-\ell}$ acts on the set $(**)$. Hence a function on

$$\{(\tau_1,\cdots,\tau_r) \mid 0 < \tau_1 < \tau_2 < \cdots < \tau_r,\quad \tau_i \neq 1\ (i=1,\cdots,r)\}$$

is uniquely extensible to a H-invariant function on X'.

Putting $\mu = p+q-2r$, we define a H-invariant open subset X_1 of X as follows:

$X_1 = \{ j \in X \mid$ the multiplicity of eigenvalue 1 in the matrix $\tilde{\sigma}(j)$ is at most $\mu+2\}$.

Then $j \in \bigcup_{\ell=0}^{r} J_\ell$ belongs to X_1 iff $\tau_i = 1$ holds for at most one i in the τ-parameter $(\tau_1,\cdots,\tau_r)$ of j.

Put $\omega = \prod_{r \geqq i > j \geqq 1} (\tau_i - \tau_j)$, $L_i = 4\tau_i(\tau_i - 1)\partial^2/\partial\tau_i^2 + 4\{(\mu+2)\tau_i - 1\}\partial/\partial\tau_i$ $(i=1,\cdots,r)$ and $\mathcal{S} = \{\omega^{-1} S(L_1,\cdots,L_r)\omega \mid$ S is a symmetric polynomial with r-variables$\}$. Due to Hoogenboom [5], we have the following

<u>Lemma</u> There exists an isomorphism $\mathfrak{I}$ of $\mathbb{D}(X)$ onto $\mathcal{S}$ such that, for any $D \in \mathbb{D}(X)$, $f \in C^\infty(X)$ and $\ell = 0,1,\cdots,r$, we have

$$\mathfrak{I}(D)|_{J_\ell'}(f|_{J_\ell'}) = (Df)|_{J_\ell'}.$$

<u>Definition</u> Let D_k be the element of $\mathbb{D}(X)$ defined by the condition

$$\mathfrak{I}(D_k) = \omega^{-1}(L_1^k + L_2^k + \cdots + L_r^k)\,\omega.$$

Due to the lemma above, $D_1, D_2, \cdots, D_r$ form a system of free generators of the commutative algebra $\mathbb{D}(X)$. Hence we have the bijection $\chi \mapsto (\lambda_1,\cdots,\lambda_r)$ of the set of the characters of $\mathbb{D}(X)$ onto the set of non-ordered r-tiples of complex numbers given by

$$\chi(D_k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_r^k \quad (k = 1,2,\cdots,r).$$

Let $\chi_{\lambda_1,\cdots,\lambda_r}$ denote the character $\chi : \mathbb{D}(X) \to \mathbb{C}$ defined by the above formulae.

Definition Let $\chi=\chi_{\lambda_1,\cdots,\lambda_r}$ be a character of $\mathbb{D}(X)$. In the case of $\lambda_i\neq\lambda_j (i\neq j)$, χ is called regular. Otherwise, χ is called singular. When $\lambda_1=\lambda_2=\cdots=\lambda_r$ holds, χ is said to be most singular.

Let Θ be an I.E.D. on X' with the infinitesimal character $\chi_{\lambda_1,\cdots,\lambda_r}$, and Π_ℓ the restriction of Θ to J'_ℓ -then $u=\omega\Pi_\ell$ satisfies the system of differential equations

$$(L_1^k+L_2^k+\cdots+L_r^k)u=(\lambda_1^k+\lambda_2^k+\cdots+\lambda_r^k)u \qquad (k=1,2,\cdots,r).$$

§3. Main results

Keep the notations in the previous sections. We set $\rho=\mu+1=p+q-2r+1$, $\lambda(s)=s^2-\rho^2$ and

$$\Lambda_d=\{\lambda(s)\mid s=\pm(\rho+2i),\ i=0,1,2,\cdots\}=\{4i(i+\mu+1)\mid i=0,1,2,\cdots\}$$

$$\Lambda_s=\{\lambda(s)\mid s=\pm(\rho+2i),\ i=-1,-2,\cdots,-\mu\}=\{4i(i+\mu+1)\mid i=-1,\cdots,-\mu\}.$$

We write $k_i=k(i)$ for $k\in\mathfrak{S}_r$. Put $L=4\tau(\tau-1)d^2/d\tau^2+4\{(\mu+2)\tau-1\}d/d\tau$. Let $F(\tau,\lambda)=F_1(\tau,\lambda)$ denote the real analytic function in $\tau>0$ satisfying the differential equation $(L-\lambda)F=0$ and $F(1,\lambda)=1$. The function $G(\tau,\lambda)=F(1-\tau,\lambda)$ is a hypergeometric function. We note that $\lambda\in\Lambda_d$ holds if and only if $F(\tau,\lambda)$ is real analytic at $\tau=0$. Furthermore we choose, once for all, a series of functions $F_i(\tau,\lambda)$ $(i=2,3,\cdots)$ of τ satisfying the condition $(L-\lambda)F_i=F_{i-1}$.

We will describe our results, dividing into two subsections, according as χ is regular or singular.

§§3.1. The case that $\chi=\chi_{\lambda_1,\cdots,\lambda_r}$ is regular

Put $\mathfrak{S}_{r,\ell}=\{k\in\mathfrak{S}_r\mid \lambda_{k_i}\in\Lambda_d\ (1\leqq i\leqq\ell)\}$. Now, as a complete system R_ℓ of representatives of $\mathfrak{S}_{r,\ell}/(\mathfrak{S}_\ell\times\mathfrak{S}_{r-\ell})$, we take

$$R_\ell=\{k\in\mathfrak{S}_{r,\ell}\mid k_1<k_2<\cdots<k_\ell,\ k_{\ell+1}<k_{\ell+2}<\cdots<k_r\}.$$

For any $\nu_1,\cdots,\nu_s\in\mathbb{C}$, put

$$D_{\nu_1,\cdots,\nu_s}(x_1,\cdots,x_s)=\det\begin{bmatrix}F(x_1,\nu_1) & \cdots\cdots & F(x_s,\nu_1)\\ \vdots & & \vdots\\ F(x_1,\nu_s) & \cdots\cdots & F(x_s,\nu_s)\end{bmatrix}.$$

Let Π be a function on $\bigcup_{\ell=0}^{r}J'_\ell$, and let Π_ℓ denote the restricton of Π to J'_ℓ. We will consider the following three conditions with respective to $\Pi=(\Pi_\ell)_{\ell=0,1,\cdots,r}$.

(1) For any $\ell\in\{0,1,2,\cdots,r\}$,

$$\omega\Pi_\ell(\tau_1,\cdots,\tau_r)=\sum_{k\in R_\ell}{}_\ell C_k\, D_{\lambda_{k_1},\cdots,\lambda_{k_\ell}}(\tau_1,\cdots,\tau_\ell)\times D_{\lambda_{k_{\ell+1}},\cdots,\lambda_{k_r}}(\tau_{\ell+1},\cdots,\tau_r).$$

with some constants ${}_\ell C_k$. (When $R_\ell=\phi$ i.e. $\ell>{}^\#[\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\cap\Lambda_d]$, we put $\Pi_\ell=0$.)

(2) $\Pi=0$.

(3) There exists a $\Theta\in\mathcal{D}'_{\chi,H}(X)$ such that Π is the restriction of Θ to $\bigcup_{\ell=0}^{r}J'_\ell$.

Theorem 1. Retain the above notations. Suppose that $\chi=\chi_{\lambda_1,\cdots,\lambda_r}$ is regular. Then, for $X = U(p,q)/\{(U(r)\times U(p-r,q)\}$ with $2r\leqq p$ and $1\leqq r\leqq q$, we obtain that

(a) the condition (3) implies the condition (1), in the case where $\Lambda_s\cap\{\lambda_1,\lambda_2,\cdots,\lambda_r\}=\phi$ and

(b) the condition (3) implies the condition (2), in the case where $\Lambda_s\cap\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\neq\phi$.

Let us consider a condition weaker than (3):

(4) There exists a $\Theta\in\mathcal{B}'_{\chi,H}(X_1)$ such that Π is the restriction of Θ to $\bigcup_{\ell=0}^{r} J'_\ell$.

Then the statement of Theorem 1 can be expressed in more detail as follows:

Proposition 2. Suppose $\chi=\chi_{\lambda_1,\cdots,\lambda_r}$ is regular. Then, the condition (4) is equivalent to the condition (1) [resp.(2)], in case that $\Lambda_s\cap\{\lambda_1,\lambda_2,\cdots,\lambda_r\}=\phi$ [resp. $\Lambda_s\cap\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\neq\phi$].

Moreover, we assume that $\chi=\chi_{\lambda_1,\cdots,\lambda_r}$ is generic. Note that $\Theta|_{J'_\ell}=0$ $(\ell\neq 0)$ for $\Theta\in\mathcal{B}'_{\chi,H}(X)$, because of $\lambda_i\notin\Lambda_d$ $(i=1,2,\cdots,r)$. Hence, using the undermentioned Proposition 3 owing to Oshima, we see that the inverse of Theorem 1 follows (Theorem 4).

Proposition 3. If χ is generic, then there is a $\Theta\in\mathcal{B}'_{\chi,H}(X)$ satisfying the condition $(\mathrm{supp}\,\Theta)\cap J'_0\neq\phi$.

Theorem 4. Let Π be a (real analytic) function on $\bigcup_{\ell=0}^{r} J'_\ell$. We put $\Pi_\ell=\Pi|_{J'_\ell}$. Suppose that $\chi=\chi_{\lambda_1,\cdots,\lambda_r}$ is generic. Then, if and only if Π is the restriction of a $\Theta\in\mathcal{B}'_{\chi,H}(X)$ to $\bigcup_{\ell=0}^{r} J'_\ell$, we get

(a) $\Pi_\ell=0$ $(1\leqq\ell\leqq r)$,

(b) $\omega\Pi_0 = c\det(F(\tau_i,\lambda_j))_{1\leqq i,j\leqq r}$ for some $c\in\mathbb{C}$.

At present, we conjecture that, even when χ is not generic, the inverse of Theorem 1 holds.

§§3.2. The case that χ is singular

For simplicity, we confine ourselves to the the case that χ is most singular, say, $\chi=\chi_{\lambda,\lambda,\cdots,\lambda}$ for some $\lambda\in\mathbb{C}$. For a map $j:\{1,\cdots,r\}\to\{1,\cdots,r\}$, put $j_i=j(i)$. We set

$$D^j(x_1,x_2,\cdots,x_r;\lambda)=D^{j_1,\cdots,j_r}(x_1,x_2,\cdots,x_r;\lambda)$$
$$=\det\begin{bmatrix} F_{j_1}(x_1,\lambda) & F_{j_2}(x_2,\lambda) & \cdots\cdots & F_{j_r}(x_r,\lambda) \\ F_{j_1-1}(x_1,\lambda) & F_{j_2-1}(x_2,\lambda) & \cdots\cdots & F_{j_r-1}(x_r,\lambda) \\ \vdots & \vdots & & \vdots \\ F_{j_1-r+1}(x_1,\lambda) & F_{j_2-r+1}(x_2,\lambda) & \cdots\cdots & F_{j_r-r+1}(x_r,\lambda) \end{bmatrix}.$$

Put $S=\{j:\{1,\cdots,r\}\to\{1,\cdots,r\}\mid j_\alpha\geqq\alpha\ (\alpha=1,\cdots,r)\}$.

Proposition 5. Let V_λ be the space of the solutions (around $(\tau_1,\cdots,\tau_r) = (1,\cdots,1)$) of a system of differential equations

(#) $\qquad (L_1^k+L_2^k+\cdots+L_r^k)u = r\lambda^k u \quad (k=1,\cdots,r)$

which are real analytic. Then, as a basis of V_λ, we can take $\{D^j(\tau_1,\cdots,\tau_r;\lambda) \mid j \in S\}$.

For $\Theta \in \mathcal{D}'_{\chi_{\lambda,\lambda,\cdots,\lambda},H}(X)$, $u = \omega\Theta|_{\mathfrak{l}}$ satisfies (#) and is extensible to a function real analytic at $(\tau_1,\cdots,\tau_r) = (1,\cdots,1)$. Adding the "connection conditions" of the I.E.D. at the points of X_1-X' and H-invariance of the I.E.D. to Proposition 5, we obtain the following theorem.

Theorem 6. Suppose that χ is most singular, say, $\chi=\chi_{\lambda,\lambda,\cdots,\lambda}$. For $\Theta \in \mathcal{D}'_{\chi,H}(X)$, we put $\Pi_\ell=\Theta|_{\mathfrak{l}}$.

(i) For some constants ${}_\ell C_j$, we have

$$\omega\Pi_\ell(\tau_1,\cdots,\tau_r) = \sum_{j\in S_\ell} {}_\ell C_j \Big(\sum_{\sigma\in\mathfrak{S}_\ell\times\mathfrak{S}_{r-\ell}} \operatorname{sgn}\sigma\ D^j(\tau_{\sigma(1)},\cdots,\tau_{\sigma(r)};\lambda)\Big) \quad (\ell = 0,1,\cdots,r),$$

where we set $F_i(\tau,\lambda) = 0$ for $i= 0,-1,-2,\cdots$ and

$S_\ell = \{j: \{1,\cdots,r\} \to \{1,\cdots,r\} \mid j_1 \geqq j_2 \geqq \cdots \geqq j_\ell,\ j_{\ell+1}=j_{\ell+2}=\cdots=j_r= r\}$.

(ii) In particular, when $\lambda \notin \Lambda_d$, it follows that

(a) $\Pi_\ell = 0 \quad (\ell=1,2,\cdots,r)$,

(b) $\omega\Pi_0(\tau_1,\cdots,\tau_r) = c\, D^{r,r,\cdots,r}(\tau_1,\cdots,\tau_r)$ for some $c \in \mathbb{C}$.

Moreover, when $\lambda \in \Lambda_s$, we have $\Pi_0 = 0$.

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