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Oscillator representations and a super dual pair

Kyo NISHIYAMA

Faculty of Science and Engeneering, Tokyo Denki University

Introduction. The aim of this report is to exhibit fruiteful structures of representations of orthosymplectic Lie super algebras. Of course for theoretical physists Lie super algebras are indispensable tools, among them orthosymplectic algebras are of particular importance (for example, see [1], [2], [3], [10]). However, it seems that mathematicians including "representatians" are less interested in the theory of Lie super algebras. Even R.Howe would not use the notion of Lie super algebras in spite of his excellent works on invariant theory of commuting and anti-commuting variables (see, e.g., [5]).

Recently the author was aware of the importance of the role of Heisenberg super algebra in the unified theory of Weil representations (commuting variables) for symplectic algebras and spin representations (anti-commuting variables) for orthogonal algebras. Analysis of Weil representations requires the use of Heisenberg algebras ([6], [7], [11], [12] and so on). To extend the theory to spin representations, we need Heisenberg super algebras! Classically this is achieved by Dirac spinors (see, e.g., [9, §1]), and Heisenberg super algebra represents Dirac spinors with multiplications or bracket product. To his regret, this short lecture does not allow the author to give a full landscape of the topics. For detailed discussions, we refer the readers to [8].

Here we start with a result of [8], namely oscillator representations of osp(n, m; $\mathbb{R}$). After preparing some basic notions in §1, we review the definition of oscillator representations in §§2 and 3 (Theorem 3.2). The heart of this report consists in §4, where a most simple example of 'super' dual pairs in osp(2n, n; $\mathbb{R}$) is given. We treat a pair osp(2, 1; $\mathbb{R}$)$\times$so(n). It seems strange that a 'super' dual pair is a pair of a Lie super algebra and a usual Lie algbera. To settle down this unease feeling completely, it is necessary to understand unitary representations of Lie super algebras well. We postpone the problem to the future study.

For the pair osp(2, 1; $\mathbb{R}$)$\times$so(n), we study the decomposition of the oscillator representation and solve the problem completely (Theorem 4.6). The way of the decomposition produces interesting examples of analysis of commuting and anti-commuting variables. It also reproduces a part of the results of [4] (Corollary 4.5). In the last part we remark about Howe's conjecture for super algebra case.

§1. Lie super algebras.

1.1. Definition and basic notions. Let $K=\mathbb{R}$ or $\mathbb{C}$ (field of real numbers or complex numbers). We put $\varepsilon(\alpha,\beta)=(-1)^{\alpha\beta}$ for $\alpha,\beta\in Z_2$.

Definition 1.1. A Z_2-graded (non-associative) algebra $L=L_0\oplus L_1$ over K with multiplication $[\cdot,\cdot]$ is called a Lie super algebra (=LSA) if the bracket product $[\cdot,\cdot]$ satisfies the followings:

(1.1) $\quad [L_\alpha, L_\beta]\subseteq L_{\alpha+\beta} \qquad (\alpha,\beta\in Z_2)$,

(1.2) $\quad [A,B]=-\varepsilon(\alpha,\beta)[B,A] \qquad (A\in L_\alpha, B\in L_\beta)$,

(1.3) $\quad \varepsilon(\gamma,\alpha)[A,[B,C]]+\varepsilon(\alpha,\beta)[B,[C,A]]+\varepsilon(\beta,\gamma)[C,[A,B]]=0$

$\qquad (A\in L_\alpha, B\in L_\beta, C\in L_\gamma)$.

The condition (1.1) requires nothing but that L is a Z_2-graded algebra. The conditions (1.2) and (1.3) are called super skew symmetry and super Jacobi identity respectively.

Let A be a Z_2-graded associative algebra so that A is a direct sum of homogeneous subspaces $A = A_0 \oplus A_1$ and the multiplication is compatible with the grading: $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ $(\alpha, \beta \in Z_2)$. If we put

$$[x, y] = xy - \varepsilon(\xi, \eta)yx \qquad (x \in A_\xi, y \in A_\eta),$$

then with this bracket product, A is an LSA. Only thing to check is super Jacobi identity (1.3), but this is easily verified after some calculations. We always considered a Z_2-graded associative algebra as an LSA in this fashion.

Let $V = V_0 \oplus V_1$ be a Z_2-graded vector space. Then the endomorphism algebra gl(V) of V has natural Z_2-graded algebra structure:

$$gl(V)_\alpha = \{A \in gl(V) \mid AV_\gamma \subseteq V_{\gamma+\alpha} \ (\gamma \in Z_2)\}.$$

Since gl(V) is a Z_2-graded associative algebra, it is an LSA by the above manner. We denote this LSA by $gl(V; \varepsilon)$ or simply by gl(V) if there is no confusion.

Definition 1.2. Let L be an LSA and V a Z_2-graded vector space. We call a homomorphism $\rho: L \to gl(V; \varepsilon)$ a representation of L.

§2. Orthosymplectic algebra in Clifford-Weyl algebra.

2.1. Definition of Clifford-Weyl algebra. Let $V = V_0 \oplus V_1$ be a Z_2-graded vector space and b a bilinear form on V such that

$$b(x, y) = -\varepsilon(\xi, \eta)b(y, x) \qquad (x \in V_\xi, y \in V_\eta),$$

where $\varepsilon(\xi, \eta) = (-1)^{\xi\eta}$. We call b a super skew symmetric bilinear form. Moreover we assume that b is homogeneous of degree zero, i.e.,

$$b(x, y) \neq 0 \ (x \in V_\xi, y \in V_\eta) \text{ implies } \xi + \eta = 0.$$

Let $T(V)$ be a tensor algebra of V which is Z_2-graded. Consider a bi-ideal $I(b)$ of $T(V)$ generated by the elements $x \otimes y - \varepsilon(\xi, \eta) y \otimes x - b(x, y)$, where $x \in V_\xi$ and $y \in V_\eta$. Then $I(b)$ is a homogeneous ideal. In fact, the term $x \otimes y - \varepsilon(\xi, \eta) y \otimes x$ is of degree $\xi + \eta$ and $b(x, y)$ does not vanish if and only if $\xi + \eta = 0$ by the assumption that b is homogeneous of degree zero.

Definition 2.1. We put $C = C(V; b) = T(V)/I(b)$ and call it a Clifford-Weyl algebra for (V, b). The projection $T(V) \to C$ is denoted by p.

Remark. If $V_1 = (0)$ (respectively $V_0 = (0)$), then $C(V; b)$ is a Clifford algebra (respectively Weyl algebra) for a non-degenerate bilinear form b.

2.2. Orthosymplectic algebra. Let (V, b) be as above. We say that a homogeneous linear transformation $A \in gl(V)_a$ preserves b if it satisfies

$$b(Ax, y) + \varepsilon(a, \xi) b(x, Ay) = 0 \qquad (x \in V_\xi, y \in V_\eta). \tag{2.1}$$

We denote by $osp(b)$ all the sum of homogeneous linear transformations which preserves b:

$$osp(b) = osp(b)_0 \oplus osp(b)_1,$$

$$osp(b)_a = \{A \in gl(V)_a \mid A \text{ satisfies } (2.1)\}.$$

By the bracket product defined in §1, $osp(b)$ is a sub LSA of $gl(V)$ and is called an orthosymplectic algebra with respect to b.

2.3. Embedding of orthosymplectic algebra into Clifford-Weyl algebra. For homogeneous elements $x \in V_\xi$ and $y \in V_\eta$, we put $m(x, y) = x \otimes y + \varepsilon(\xi, \eta) y \otimes x \in T_2(V)$. We define $L(b) \subseteq C(V; b)$ by

$$L(b) = \langle p(m(x, y)) \mid x, y: \text{homogeneous in } V \rangle / K\text{-vector space}.$$

Lemma 2.2. $L(b)$ *is a sub LSA of* $C(V; b)$ *(the bracket is considered to be the usual one).*

Proof consists of routine calculations. Q.E.D.

Since $V \simeq p(V) \subseteq C(V; b)$, we consider V as a subspace of $C(V; b)$. An element x of $C(V; b)$ acts on itself as an inner derivation : $\operatorname{ad} x = [x, \cdot]$. We call this representation the adjoint representation of $C(V; b)$.

Lemma 2.3. *The adjoint representation restricted to* $L(b)$ *preserves* V. *Therefore we get a representation of* $L(b)$ *on* V.

Proof. We write $m(x, y)$ instead of $p(m(x, y))$ for the sake of simplicity. For a homogeneous $v \in V_\nu$, we have

$$[m(x, y), v] = 2[xy, v] = 2(xyv - \varepsilon(\xi+\eta, \nu)vxy)$$
$$= 2(b(y, v)x + \varepsilon(\xi, \eta)b(x, v)y),$$

where $x \in V_\xi, y \in V_\eta$. Q.E.D.

We gather the above results and get

Theorem 2.4([8, Theorem 3.5]). *The adjoint representation of* $L(b)$ *on* V *gives a homomorphism of* $L(b)$ *into* $\mathrm{osp}(b)$. *Moreover, if* b *is non-degenerate, then the representation gives an isomorphism between* $L(b)$ *and* $\mathrm{osp}(b)$.

Sketch of Proof. At first we must verify that $\operatorname{ad} L(b)$ preserves b. This is a consequence of some tiresome calculations.

Next we assume that b is non-degenerate. Then both $L(b)$ and $\mathrm{osp}(b)$ have the same dimension, we conclude that $\operatorname{ad}|_{L(b)}$ on V is faithful and an isomorphism between $L(b)$ and $\mathrm{osp}(b)$. Q.E.D.

Due to this theorem, we get a representation $\bar{\tau}$ of $\mathrm{osp}(b)$ from a representation τ of $C(V; b)$ by restricting τ to a sub LSA $L(b) \simeq \mathrm{osp}(b)$.

§3. Oscillator representation for orthosymplectic algebra.

In this section we only treat superalgebras over $K = \mathbb{R}$. All the representations in this section are representations on the complex vector spaces.

We assume that the super skew symmetric bilinear form b on $V=V_0 \oplus V_1$ is non-degenerate and there are basis $\{p_k, q_k | 1 \leqq k \leqq n\}$ of V_0 and $\{r_\ell, s_\ell | 1 \leqq \ell \leqq m\} \cup \{c\}$ of V_1 (c appears if and only if $\dim V_1$ is odd) such that

$$b(p_i, q_j) = -b(q_j, p_i) = \delta_{ij}, \quad b(p_i, p_j) = b(q_i, q_j) = 0,$$

and $\{r_\ell, s_\ell | 1 \leqq \ell \leqq m\} \cup \{c\}$ is an orthogonal basis in V_1 with respect to b with length $\sqrt{2}$.

Clifford-Weyl algebra $C(V; b)$ is then generated by p, q, r, s and c with relations:

$$p_i q_j - q_j p_i = \delta_{ij}, \; r_i s_j + s_j r_i = 0, \; r_i r_j + r_j r_i = 2\delta_{ij}, \; s_i s_j + s_j s_i = 2\delta_{ij},$$

and all the other pairs of p, q, r, s commute with each other. In addition to these, if $\dim V_1$ is odd, there are relations which contain c:

$c^2 = 1$, $cr_i + r_i c = 0$, $cs_i + s_i c = 0$, c commutes with p_k and q_k.

Definition 3.1. A representation (ρ, E) of an LSA L is called super unitary if there exists a super Hermitian form (,) on E such that

$$(3.1) \qquad (\rho(x)v, w) = -(-1)^{\xi\nu}(v, \rho(x)w) \qquad (x \in L_\xi, v \in E_\nu \text{ and } w \in E_\omega)$$

and

$$(3.2) \qquad (v, w) = 0 \text{ if } \nu + \omega = 1,$$

$$(v, v) > 0 \text{ for } v \in E_0 \setminus \{0\}, \quad \delta\sqrt{-1}(w, w) > 0 \text{ for } w \in E_1 \setminus \{0\}.$$

Here $\delta = \pm 1$ is called an associated constant for (ρ, E) (see [4]).

Let $C(r_\ell, c \,|\, 1 \leqq \ell \leqq m)$ (respectively $C(r_\ell \,|\, 1 \leqq \ell \leqq m)$) be a subalgebra of $C(V; b)$ which is generated by the elements $\{r_\ell \,|\, 1 \leqq \ell \leqq m\} \cup \{c\}$ (respectively $\{r_\ell \,|\, 1 \leqq \ell \leqq m\}$). Then $C(r_\ell, c \,|\, 1 \leqq \ell \leqq m)$ is a usual Clifford algebra with usual Z_2-grading $C^0(r_\ell, c \,|\, 1 \leqq \ell \leqq m)$ and $C^1(r_\ell, c \,|\, 1 \leqq \ell \leqq m)$. We denote by α_ℓ an automorphism of $C(r_\ell, c \,|\, 1 \leqq \ell \leqq m)$ such that

$$\alpha_\ell(r_i) = \begin{cases} r_i & \text{if } i = \ell \\ -r_\ell & \text{if } i = \ell \end{cases}, \qquad \alpha_\ell(c) = c.$$

Theorem 3.2([8, Theorem 5.5]). *The representation* $(\tilde{\rho}, \mathrm{F})$ *of* osp(b; $\mathbb{R}$) *obtained from the representation* (ρ, F) *of* C(V; b) *given below is super unitary:*

$$\mathrm{F} = \mathrm{F}_0 \oplus \mathrm{F}_1 ,$$
$$\mathrm{F}_0 = \mathbb{C}[z_k \mid 1 \leqq k \leqq n] \otimes C^0{}_C(r_\ell, c \mid 1 \leqq \ell \leqq m),$$
$$\mathrm{F}_1 = \mathbb{C}[z_k \mid 1 \leqq k \leqq n] \otimes C^1{}_C(r_\ell, c \mid 1 \leqq \ell \leqq m),$$

and the operators are given by

$$\rho(p_k)(f(z)\otimes v) = \sqrt{-1}\, j\left(\frac{1}{\sqrt{2}}(z_k - \frac{\partial}{\partial z_k})f(z)\right)\otimes v ,$$

$$\rho(q_k)(f(z)\otimes v) = j\left(\frac{1}{\sqrt{2}}(z_k + \frac{\partial}{\partial z_k})f(z)\right)\otimes v ,$$

$$\rho(r_\ell)(f(z)\otimes v) = f(z)\otimes r_\ell v , \tag{3.3}$$

$$\rho(s_\ell)(f(z)\otimes v) = f(z)\otimes \sqrt{-1}\, r_\ell \alpha_\ell(v) ,$$

$$\rho(c)(f(z)\otimes v) = f(z)\otimes cv ,$$

where j *is a square root of* $\sqrt{-1}$ *and* c *does not appear if* $\dim V_1$ *is even.*

We call this representation $(\tilde{\rho}, \mathrm{F})$ oscillator representation. Remark that this representation is not irreducible and has two irreducible components.

§4. Super dual pair.

4.1. After this section we assume that $n = 2m$ and $\dim V_1 = 2m$ is even. Then we write osp(2n, n; $\mathbb{R}$) = osp(b; $\mathbb{R}$). Put

$$c_{2j-1} = r_j , \quad c_{2j} = s_j \qquad (1 \leqq j \leqq m).$$

By Theorem 2.4, we identify osp(2n, n; $\mathbb{R}$) with a sub LSA of C(V; b). Hence m(x, y) in §2 is an element of osp(2n, n; $\mathbb{R}$). Put

$$P = \sum_{i=1}^{n} m(p_i, c_i) ; \quad Q = \sum_{i=1}^{n} m(q_i, c_i) .$$

Then we have

$$[P,Q]=2\sum_{i=1}^{n} m(p_i,q_i)\ ;\ [P,P]=2\sum_{i=1}^{n} m(p_i,p_i)\ ;\ [Q,Q]=2\sum_{i=1}^{n} m(q_i,q_i)\ ,$$

and

$\underline{a}$ $= <P, Q>$/generated as an LSA

$= <P, Q, [P, Q], [P, P], [Q, Q]>$/generated as a vector space over $\mathbf{R}$

turns to be isomorphic to $\mathrm{osp}(2,1;\mathbf{R})$.

Proposition 4.1. *Let* $\underline{a}'$ *be a commutant of* $\underline{a}\simeq\mathrm{osp}(2,1;\mathbf{R})$. *Then* $\underline{a}'$ *is a Lie algebra isomorphic to* $\mathrm{so}(n)$, *and* $\underline{a}$ *and* $\underline{a}'$ *are commutants with each other.*

More explicitly, $\underline{a}'$ has a basis

$$m(p_j, q_i)-m(p_i, q_j)+\tfrac{1}{2}m(c_i, c_j) \qquad (1\leqq i\neq j\leqq m).$$

In the matrix form, $\underline{a}$ and $\underline{a}'$ are of the following forms:

$$a=\left\{\begin{pmatrix} a1_n & b1_n & d1_n \\ c1_n & -a1_n & e1_n \\ -e1_n & d1_n & 0_n \end{pmatrix} \,\middle|\, a,b,c,d,e\in\mathbf{R}\right\},\quad a'=\left\{\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} \,\middle|\, A\in\mathrm{so}(n)\right\}.$$

4.2. Decomposition of the oscillator representation and Howe's conjecture. How does the oscillator representation break up as a representation of $\underline{a}\times\underline{a}'$? Lie algebra case tells us that the decomposition should be multiplicity free (for example, see [6, Theorem D]) and, in fact, that is true! However, in the super case, Howe's conjecture is false. We start with the decomposition of the representation of $\underline{a}'\simeq\mathrm{so}(n)$. We write oscillator representation $(\bar{\rho}, F)$ instead of $(\bar{\rho}, F)$ for the sake of simplicity. Since $\bar{\rho}$ is a restriction of the representation ρ of $C(V; b)$, there is no problem to do so.

Proposition 4.2. *Let* (ρ, F) *be the oscillator representation of* $\mathrm{osp}(2n, n; \mathbf{R})$ *considered above. Put*

$$PF(k)=(\textit{homogeneous polynomials of degree } k)\otimes C(r_\ell \mid 1\leqq \ell\leqq m)\subseteq F.$$

(1) *The space* $PF(k)$ *is invariant under the action of* $\underline{a}'$ *and it breakes up as* $\underline{a}'$*-module like this; for* $k\geqq 1$,

$$PF(k)\simeq PF(k-2)\oplus\tau_1^{+}(k)\oplus\tau_1^{-}(k)\oplus\tau_2^{+}(k)\oplus\tau_2^{-}(k)\ , \tag{4.1}$$

where $\tau_j^{\pm}$ *is an irreducible finite dimensional representation of* so(n) *with highest weight listed below in* (2). *Intertwining map* PF(k−2)→PF(k) *is given by the multiplication by* $(z_1^2+z_2^2+\cdots+z_n^2)\otimes 1$.

(1') $PF(0)=C(r_\ell \mid 1\leqq \ell \leqq m)\simeq \tau_1^+(0)\oplus\tau_1^-(0)$.

(2) *Heighest weights and heighest weight vectors for* $\tau_j^{\pm}$ *are explicitly given as follows.*

	highest weight	highest weight vectors
τ_1^+	$(k+\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})$	$v_k^+=(z_1+\sqrt{-1}\,z_2)^k\otimes 1$
τ_1^-	$(k+\frac{1}{2},\frac{1}{2},\cdots,-\frac{1}{2})$	$v_k^-=(z_1+\sqrt{-1}\,z_2)^k\otimes r_m$
τ_2^+	$(k-\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})$	$w_k^+=(z_1+\sqrt{-1}\,z_2)^{k-1}\{\Sigma_{1\leq j\leq m-1}(z_{2j-1}+\sqrt{-1}\,z_{2j})\otimes r_j r_m$ $+(z_{n-1}-\sqrt{-1}\,z_n)^k\otimes 1\}$
τ_2^-	$(k-\frac{1}{2},\frac{1}{2},\cdots,-\frac{1}{2})$	$w_k^-=(z_1+\sqrt{-1}\,z_2)^{k-1}\{\Sigma_{1\leq j\leq m}(z_{2j-1}+\sqrt{-1}\,z_{2j})\otimes r_j\}$

Table 4.3.

Sketch of Proof. Since operators $(z_1^2+z_2^2+\cdots+z_n^2)\otimes 1$ and $\rho(\underline{a}')$ commute, the map $f\otimes v \to (z_1^2+z_2^2+\cdots+z_n^2)f\otimes v$ gives an embedding $PF(k-2)\hookrightarrow PF(k)$. Let us see that the vectors listed in Table 4.3 are really highest weight vectors. Making use of the explicit formula (3.3) for ρ, we see positive simple root vectors of $\underline{a}'$ act on F as

$$\Big\{(z_{2j-1}+\sqrt{-1}\,z_{2j})\Big(\frac{\partial}{\partial z_{2j+1}}\pm\sqrt{-1}\frac{\partial}{\partial z_{2j+2}}\Big)-$$

$$-(z_{2j+1}\pm\sqrt{-1}\,z_{2j+2})\Big(\frac{\partial}{\partial z_{2j-1}}+\sqrt{-1}\frac{\partial}{\partial z_{2j}}\Big)\Big\}\otimes 1+$$

$$+1\otimes r_j r_{j+1}(1-\alpha_j)(1\mp\alpha_{j+1}) \qquad (1\leqq j\leqq m-1).$$

Now one can check that the vectors in Table 4.3 are killed by the operators above. Similarly their weights can be calculated and we conclude that they are highest weight vectors. Since dimensions of the both hand sides of (4.1) are equal by Weyl's dimension formula, we obtain desired decomposition. Q.E.D.

Next we consider the decomposition of the oscillator representation ρ as a representation of $\underline{a} \simeq osp(2, 1; \mathbb{R})$. Since $\underline{a}$ and $\underline{a}'$ are commutants with each other, $\rho(\underline{a})$ preserves a space of highest weight vectors of $\underline{a}'$ which have a fixed highest weight. Let $v_k^{\pm}$ and $w_k^{\pm}$ be highest weight vectors as in Table 4.3. Then the space

$$V^+(k) = \{f(z_1^2 + z_2^2 + \cdots + z_n^2)v_k^+, f(z_1^2 + z_2^2 + \cdots + z_n^2)w_{k+1}^+ \mid f(x) \in \mathbb{C}[x]\}$$

consists of all the highest weight vecors with weight $(k+\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and

$$V^-(k) = \{f(z_1^2 + z_2^2 + \cdots + z_n^2)v_k^-, f(z_1^2 + z_2^2 + \cdots + z_n^2)w_{k+1}^- \mid f(x) \in \mathbb{C}[x]\}$$

is the space of all the highest weight vectors with weight $(k+\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2})$.

Proposition 4.4. (1) $V^{\pm}(k)$ *is an irreducible lowest weight module of* $\underline{a} \simeq osp(2, 1; \mathbb{R})$ *with lowest weight* $k+\frac{1}{2}$. *As a representation of* $\underline{a}_0 \simeq sl(2; \mathbb{R})$ (*even part of* $\underline{a}$), $V^{\pm}(k)$ *is a direct sum of two holomorphic discrete series representations with lowest weights* $k+\frac{1}{2}$ *and* $k+3/2$.

(2) *A lowest weight vector for* $V^{\pm}(k)$ *is* $v_k^{\pm}$. *For* $\underline{a}_0 \simeq sl(2; \mathbb{R})$, *lowest weight vectors are* $v_k^{\pm}$ *and* $w_k^{\pm}$.

Corollary 4.5(see [4]). *An irreducible lowest weight module of* $osp(2, 1; \mathbb{R})$ *with lowest weight* $k+\frac{1}{2}$ *is unitarizable.*

Remark. All the unitary representations of $osp(2, 1; \mathbb{R})$ are classified by H.Furutsu and T.Hirai ([4, Theorem 5.11]). As a result they are either lowest weight modules or highest weight modules extended from (anti-)holomorphic discrete series representations of $osp(2, 1; \mathbb{R})_0 \simeq sl(2; \mathbb{R})$.

Proof of Proposition 4.4. Negative simple (odd) root vector of $\underline{a} \simeq osp(2, 1; \mathbb{R})$ acts on $V^{\pm}(k)$ as an operator

$$N = \sum_{j=1}^{m} \left\{ \frac{\partial}{\partial z_{2j-1}} \otimes r_j + \sqrt{-1} \frac{\partial}{\partial z_{2j}} \otimes r_j a_j \right\}.$$

Since $Nv_k^{\pm}=0$ and $Nw_k^{\pm}=(2k+n-2)v_{k-1}^{\pm}$, we conclude that $v_k^{\pm}$ are only lowest weight vectors in $V^{\pm}(k)$. The rest of the assertions are clear. Q.E.D.

We summarize above results into

Theorem 4.6. *The oscillator representation* (ρ, F) *is multiplicity free as a representation of* $\underline{a} \times \underline{a}' = \mathrm{osp}(2, 1; \mathbb{R}) \times \mathrm{so}(n)$.

(1) *The decomposition of* ρ *is given as follows:*

$$\rho \simeq \sum_{k=0}^{\infty} \left\{ D_{k+\frac{1}{2}} \otimes \tau_1^{+}(k) \oplus D_{k+\frac{1}{2}} \otimes \tau_1^{-}(k) \right\},$$

where $D_{k+\frac{1}{2}}$ *is an irreducible lowest weight module of* $\mathrm{osp}(2, 1; \mathbb{R})$ *with lowest weight* $k+\frac{1}{2}$.

(2) *For a generic vector for* $D_{k+\frac{1}{2}} \otimes \tau_1^{\pm}(k)$, *we can take* $v_k^{+} = (z_1 + \sqrt{-1}\, z_2)^k \otimes 1$ *and* $v_k^{-} = (z_1 + \sqrt{-1}\, z_2)^k \otimes r_m$ *respectively.*

From this theorem, the correspondence between irreducible unitary representations of $\mathrm{osp}(2, 1; \mathbb{R})$ and those of $\mathrm{so}(n)$ is not one-to-one but one-to-two! Howe's conjecture, which is valid for usual Lie algebra over $\mathbb{R}$, asserts that the correspondence of irreducible unitary representations through a reductive dual pair is one-to-one. So Howe's conjecture is not valid for LSA's. However, the author hopes that the conjecture is valid for super dual pairs of higher rank.

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