ユニタリ表現論セミナー報告集 VIII, 1988
pp.63-66

On monomial representations of exponential solvable Lie groups

Hidenori Fujiwara

The study of monomial representations of exponential solvable Lie groups was taken up by Dixmier and Takenouchi, and after Kirillov created the orbit method, fundamental works were done by Bernat, Pukanszky and Vergne.

Let $G=\exp\mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f\in\mathfrak{g}^*$, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ such that $f([\mathfrak{h},\mathfrak{h}])=0$ and let $H=\exp\mathfrak{h}$ be the corresponding analytic subgroup of G. One considers the monomial representation $\tau=\mathrm{ind}_H^G\chi_f$, where χ_f is the unitary character of H given by $\chi_f(\exp X)=e^{if(X)}$ $(X\in\mathfrak{h})$, and its canonical central decomposition

$$\tau=\int_{\hat{G}}^{\oplus} m_\pi\pi\, d\tilde{\mu}(\pi).$$

Now our problem is to know the measure class of $\tilde{\mu}$ and describe the multiplicity m_π in terms of orbits. About 72, Grélaud and Quint proved independently the same result as follows: if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the measure class of $\tilde{\mu}$ is the image by the Kirillov-Bernat mapping of that of a Lebesgue measure on $f+\mathfrak{h}^\perp\subset\mathfrak{g}^*$, where $\mathfrak{h}^\perp=\{\lambda\in\mathfrak{g}^*;\ \lambda|\mathfrak{h}=0\}$, and m_π is uniformly equal either to 1 or to ∞, more precisely

$$m=\begin{cases} 1 & \text{if } (f+\mathfrak{h}^\perp)\cap\Omega(\pi) \text{ is a manifold of dimension } \dim\Omega(\pi)/2;\\ \infty & \text{otherwise;}\end{cases}$$

here $\Omega(\pi)$ denotes the coadjoint orbit corresponding to $\pi\in\hat{G}$. In any way m_π is given by the number of H-orbits contained in $\Omega(\pi)\cap(f+\mathfrak{h}^\perp)$.

Moreover Quint left some conjectures. Put $Z(f+\mathfrak{h}^\perp)=\{\lambda\in f+\mathfrak{h}^\perp;\ \dim G\cdot\lambda$ is maximum amongst all orbits intersected by $f+\mathfrak{h}^\perp\}$. Let μ be a finite measure on $f+\mathfrak{h}^\perp$ equivalent to its Lebesgue measure, $p:\mathfrak{g}^*\to\hat{G}\simeq\mathfrak{g}^*/G$ the Kirillov-Bernat mapping and let $\tilde{\mu}=p_*(\mu)$ be the image of μ by p. Then his conjectures say:

(i) For all $\lambda\in Z(f+\mathfrak{h}^\perp)$, each connected component of $G\cdot\lambda\cap(f+\mathfrak{h}^\perp)$ is a differentiable manifold of dimention greater than or equal to $\dim G\cdot\lambda/2$.

(ii) $\tau=\mathrm{ind}_H^G\chi_f=\int_{\hat{G}}^{\oplus} m_\pi\pi\, d\tilde{\mu}(\pi)$ with

$$m=\begin{cases} \text{number of connected components of } \Omega(\pi)\cap(f+\mathfrak{h}^\perp) \text{ provided each} \\ \text{component is a manifold of dimension } \dim\Omega(\pi)/2;\\ \infty \quad \text{otherwise.}\end{cases}$$

Now we have fundamental results due to Benoist for symmetric case and to Corwin-Greenleaf/Grélaud for nilpotent case, which prove Quint's conjectures with only one slight modification, in (i) "all $l\in Z(f+\mathfrak{h}^{\perp})$" should be replaced by "μ almost every l in $f+\mathfrak{h}^{\perp}$", and further that multiplicity m_π is given also by the number of H-orbits included in $\Omega(\pi)\cap(f+\mathfrak{h}^{\perp})$. Finally even for exponential case we know Lipsman's works and obtain ourselves:

Theorem. The modified Quint's conjectures are true for exponential case. Furthermore we can take for multiplicuty m_π the number of H-orbits contained in $\Omega(\pi)\cap(f+\mathfrak{h}^{\perp})$.

To prove the part (i) we use essentially the Sard's theorem about the set of critical values. For the part (ii), one natural way to go is to consider a subgroup K of codimension 1 or 2 which contains H. But, when dim G/K=2, it's really a hard task to pass from K to G, as Lipsman says. So we don't follow this way and choose another well-known method. Let's recall some simple facts on groups of small dimension.

Lemma. (1) $G=\exp\mathfrak{g}_2$, $\mathfrak{g}_2=\langle X,Y\rangle_R$: $[X,Y]=Y$ (ax+b algebra). Let $f\in\mathfrak{g}_2^*$, $\mathfrak{h}=RX$, $H=\exp\mathfrak{h}$. Then $\mathrm{ind}_H^G\chi_f\simeq\mathrm{ind}_{H'}^G\chi_{Y^*}\oplus\mathrm{ind}_{H'}^G\chi_{-Y^*}$ with $H'=\exp RY$.

(2) $G=\exp\mathfrak{g}_3(\alpha)$, $\mathfrak{g}_3(\alpha)=\langle T,Y_1,Y_2\rangle_R$: $[T,Y_1]=Y_1-\alpha Y_2$, $[T,Y_2]=Y_2+\alpha Y_1$ $(\alpha\in R\backslash\{0\})$. Let $f\in\mathfrak{g}_3(\alpha)^*$, $\mathfrak{h}=RT$, $H=\exp\mathfrak{h}$. Then

$$\mathrm{ind}_H^G\chi_f\simeq\int_{[0,2\pi)}^{\oplus}\mathrm{ind}_{H'}^G\chi_{\hat\theta}\,d\theta$$

with $H'=\exp(RY_1+RY_2)$, $\hat\theta=(\cos\theta)Y^*+(\sin\theta)Y^*\in\mathfrak{g}_3(\alpha)^*$.

(3) $G=\exp\mathfrak{g}_4$, $\mathfrak{g}_4=\langle T,X,Y,Z\rangle_R$: $[X,Y]=Z$, $[T,X]=-X$, $[T,Y]=Y$ (split oscillator algebra). Let $f=\alpha T^*+\beta Z^*\in\mathfrak{g}_4^*$ $(\beta\neq0)$, $\mathfrak{h}=RT+RX+RZ$, $H=\exp\mathfrak{h}$. Then $\mathrm{ind}_H^G\chi_f\simeq\mathrm{ind}_{H'}^G\chi_f$ with $H'=\exp\mathfrak{h}'$, $\mathfrak{h}'=RT+RY+RZ$.

(4) $G=\exp\mathfrak{g}_6$, $\mathfrak{g}_6=\langle T,X_1,X_2,Y_1,Y_2\rangle_R$: $[X_i,Y_j]=\delta_{ij}Z$ $(1\leq i,j\leq2)$, $[T,Y_1]=Y_1-\alpha Y_2$, $[T,Y_2]=Y_2+\alpha Y_1$, $[T,X_1]=-X_1-\alpha X_2$, $[T,X_2]=-X_2+\alpha X_1$. Let $f=\alpha T^*+\beta Z^*$ $(\beta\neq0)$, $\mathfrak{h}=RT+RX_1+RX_2+RZ$, $H=\exp\mathfrak{h}$. Then $\mathrm{ind}_H^G\chi_f\simeq\mathrm{ind}_{H'}^G\chi_f$ with $H'=\exp\mathfrak{h}'$, $\mathfrak{h}'=RT+RY_1+RY_2+RZ$.

Now the part (ii) of the conjectures is shown by induction on dim G. If $\mathfrak{h}+[\mathfrak{g},\mathfrak{g}]\neq\mathfrak{g}$, we take an ideal $\mathfrak{g}_0$ of codimension 1 containing $\mathfrak{h}+[\mathfrak{g},\mathfrak{g}]$ and apply the induction hypothesis to $G_0=\exp\mathfrak{g}_0$. If $\mathfrak{h}+[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$, making use of a minimal non-central ideal $\mathfrak{a}$, we modify $\mathfrak{h}$. For instance, let $\mathfrak{a}$ be a minimal ideal of dimension 2. When $\mathfrak{h}\cap\mathfrak{a}=\mathfrak{a}$, we can go down to the quotient G/A, $A=\exp\mathfrak{a}$, to which is applied the induction hypothesis. Suppose that $\mathfrak{h}\cap\mathfrak{a}=\{0\}$. Let $\mathfrak{g}_0$ be the centralizer of $\mathfrak{a}$ and let $G_0=\exp\mathfrak{g}_0$. Put $\mathfrak{h}'=\mathfrak{h}\cap\mathfrak{g}_0+\mathfrak{a}$. Then $\mathfrak{h}'\subset\mathfrak{g}_0$, $\mathfrak{g}=RX\oplus\mathfrak{g}_0$ with $X\in\mathfrak{h}$ and we see

$$\mathrm{ind}_H^G\,\chi_f \simeq \int_{[0,2\pi)}^{\oplus} \mathrm{ind}_{H'}^G\,\chi_{\hat\theta}\,d\theta$$

just as in Lemma (2). Since $\mathfrak{h}'+[\mathfrak{g},\mathfrak{g}]\neq\mathfrak{g}$, we know already the decomposition of $\mathrm{ind}_{H'}^G\,\chi_{\hat\theta}$, and can compare, for any coadjoint orbit Ω, $\Omega\cap(f+\mathfrak{h}^\perp)$ with $\Omega\cap(\theta+\mathfrak{h}'^\perp)$ to obtain the multiplicities. Other cases are treated likewise.

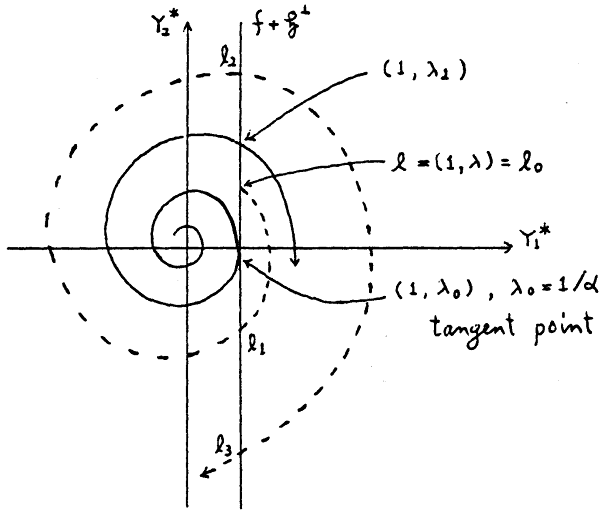
Now we leave the following problems.

1) Reciprocity: Does $\dim\,(\mathcal{H}_\pi^{-\infty})^{H,\,\chi_f\Delta_{H,G}^{1/2}}$ give the multiplicity m_π? Here $\Delta_{H,G}=\Delta_H/\Delta_G$.

2) Penney's Plancherel formula: Find $a_\pi^k\in(\mathcal{H}_\pi^{-\infty})^{H,\,\chi_f\Delta_{H,G}^{1/2}}$ $(1\leq k\leq m_\pi)$ satisfying, for all $\phi\in C_c^\infty(G)$, $\phi_H^f(e)=\int_{\hat G}\sum_{k=1}^{m_\pi}\langle\pi(\phi)a_\pi^k,a_\pi^k\rangle\,d\tilde\mu(\pi)$, where $\phi_H^f(g)=\int_H\phi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh$ with a left Haar measure dh on H.

Examples: (1) In Lemma (2), $\pi_\theta=\mathrm{ind}_{H'}^G\,\chi_{\hat\theta}$ is irreducible and we see $(\mathcal{H}_{\pi_\theta}^{-\infty})^{H,\,\chi_f\Delta_{H,G}^{1/2}}=\mathbb{C}a_\theta$ with $a_\theta:\mathcal{H}_{\pi_\theta}^{+\infty}\ni\phi\mapsto\int_H\overline{\phi(h)\chi_f(h)\Delta_{H,G}^{-1/2}}(h)dh$. So $\dim\,(\mathcal{H}_{\pi_\theta}^{-\infty})^{H,\,\chi_f\Delta_{H,G}^{1/2}}=1$ and, under proper normalizations of measures, $\phi_H^f(e)=\int_0^{2\pi}\langle\pi_\theta(\phi)a_\theta,a_\theta\rangle\,d\theta$ for all $\phi\in C_c^\infty(G)$.

(2) For the algebra in Lemma (2), we take $f=Y_1^*$ and $\mathfrak{h}=\mathbb{R}Y_1$. We introduce some notations as in the figure.



Let $\pi_\lambda=\mathrm{ind}_{H'}^G\,\chi_l$ for $\lambda_0\leq\lambda<\lambda_1$ and we take $g_n\in G$ such that $g_n\cdot l=l_n$, where $l_n=(1,y_n)\in\mathfrak{h}'^*$ with some function $y_n=y_n(\lambda)$ of λ. Then $\tau=\mathrm{ind}_H^G\,\chi_f\simeq\int_{\lambda_0}^{\lambda_1}\infty\pi_\lambda\,d\lambda$. Accordingly $a_n(\lambda):\mathcal{H}_{\pi_\lambda}^{+\infty}\ni\phi\mapsto\overline{\phi(g_n)}\in\mathbb{C}$ $(n=0,1,2,\cdots)$ belong to $(\mathcal{H}_{\pi_\lambda}^{-\infty})^{H,\,\chi_f\Delta_{H,G}^{1/2}}$ and, normalizing various measures appropriately, we have, for all $\phi\in C_c^\infty(G)$, $\phi_H^f(e)=\int_{\lambda_0}^{\lambda_1}\sum_{n=0}^{\infty}\langle\pi_\lambda(\phi)a_n'(\lambda),a_n'(\lambda)\rangle\,d\lambda$ with $a_n'(\lambda)=(1+y_n^2)^{1/2}\,|\,1-\alpha y_n\,|^{-1/2}a_n(\lambda)$.

(3) We take the split oscillator group in Lemma (3). Let $f(\alpha,\beta)=\alpha T^*+\beta Z^*$ and let $\pi(\alpha,\beta)$ the corresponding irreducible representation realized through a polarization $\mathfrak{b}=\langle T,Y,Z\rangle_{\mathbb{R}}$: $\pi(\alpha,\beta)=\mathrm{ind}_B^G\,\chi_{f(\alpha,\beta)}$, $B=\exp\mathfrak{b}$. Let's consider $\mathfrak{h}=\mathbb{R}T+\mathbb{R}Z$, $f=f(\alpha_0,\beta_0)$ with $\beta_0\neq0$. Then $\tau=\mathrm{ind}_H^G\,\chi_f\simeq\int_{\mathbb{R}}2\pi(\alpha,\beta_0)d\alpha$ and $(\mathcal{H}_{\pi(\alpha,\beta_0)})^{H,\,\chi_f\Delta_{H,G}^{1/2}}=\mathbb{C}a_1(\alpha)\oplus\mathbb{C}a_2(\alpha)$, here $a_1(\alpha):\mathcal{H}_{\pi(\alpha,\beta_0)}\ni\phi\mapsto\int_0^\infty\overline{\phi(\exp sX)s^{i(\alpha-\alpha_0)-1/2}}ds\in\mathbb{C}$, $a_2(\alpha):\mathcal{H}_{\pi(\alpha,\beta_0)}^{+\infty}\ni\phi\mapsto\int_0^\infty\overline{\phi(\exp -sX)s^{i(\alpha-\alpha_0)-1/2}}ds\in\mathbb{C}$. Finally, normalizing measures, we get our Plancherel formula for τ: $\phi_H^f(e)=\int_{\mathbb{R}}(\langle\pi(\phi)a_1(\alpha),a_1(\alpha)\rangle+\langle\pi(\phi)a_2(\alpha),a_2(\alpha)\rangle)d\alpha$ for all $\phi\in C_c^\infty(G)$.

References

[1] Y.Benoist, Analyse harmonique sur les espaces symétriques nilpotents, J. Func. Anal., 59(1984), 211-253.

[2] Y.Benoist, Multiplicité un pour les espaces symétriques exponentiels, Mém. Soc. Math. France, 15(1984), 1-37.

[3] L.Corwin, F.P.Greenleaf and G.Grélaud, Direct integral decompositions and multiplicities for induced representations of nilpotent Lie groups, Trans. Amer. Math. Soc., 304(1987), 549-583.

[4] H.Fujiwara, Représentations monomiales des groupes de Lie nilpotents, Pacific J. Math., 127(1987), 329-352.

[5] H.Fujiwara et S.Yamagami, Certaines représentations monomiales d'un groupe de Lie résoluble exponentiel, à paraître dans Advanced Studies in Pure Math..

[6] G.Grélaud, Désintégration des représentations induites des groupes de Lie résolubles exponentiels, Thèse de 3^e cycle, Univ. de Poitiers, 1973.

[7] G.Grélaud, Sur les représentations des groupes de Lie résolubles, Thèse, Univ. de Poitiers, 1984.

[8] R.Lipsman, Orbital parameters for induced and restricted representations, to appear in Trans. Amer. Math..

[9] R.Lipsman, Harmonic analysis on exponential solvable homogeneous spaces: The algebraic or symmetric cases, to appear.

[10] R.Lipsman, Induced representations of completely solvable Lie groups, to appear.

[11] S.R.Quint, Decomposition of induced representations of solvable exponential Lie groups, Dissertation, Univ. of California, Berkeley, 1973.