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On a global realization of a discrete series for SU(n,1) as applications of Szegö operator and limits of discrete series

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Abstract

In [KW] Knapp and Wallach gave an explicit imbedding of the discrete series of a connected semisimple Lie group G with finite center as a subrepresentation in the nonunitary principal series. However, it was in an infinitesimally equivalent fashion. Our aim is to give a global realization of the discrete series. Recently, when real rank of G is 1, Blank [B] gave an explicit projection operator that transfers a reducible unitary principal series onto a limit of discrete series in a global level. Therefore, applying Zuckerman's technique (see [Z]), we can shift Blank's result and construct a representation that is infinitesimally equivalent to a

discrete series of G. Then the unitarity of the representation corresponds to square-integrability of the image of a modified Szegö operator, which was conjectured in [KW]. In the case of $G=SU(n,1)$, we can obtain the square-integrability by applying the complex structure of the hermitian symmetric space G/K, and thus, get a global construction of the discrete series.

This method is completely different from ordinary one, for it starts with a limit of discrete sereis. This implies that the representations constructed by our method must be attached to a limit of discrete series, and thus they are unfortunately a part of the discrete series of G. Square-integrability of the image of the Szegö operator is still an unsettled problem except $SU(n,1)$, however, all others in this talk are valid for all real rank 1 semisimple Lie groups.

Let G be a connected semisimple Lie group with finite center and fix a maximal compact subgroup K of G. We assume that rank G = rank K, that is, G has a compact Cartan subgroup $T \subset K$. Then by Harish- Chandra [HC] this condition is equivalent with that G has a discrete series. Let $\underline{t}$ be the Lie algebra of T and W_K

the Weyl group of K. Then the set of discrete series is in bijective correspondence with the set of W_K-orbits of non-singular integral forms on $\underline{t}$. We denote by π_Λ the discrete sereis corresponding to a nonsingular integral form Λ on $\underline{t}$. Let G=ANK be an Iwasawa decomposition of G and M the centralizer of A in K; let $\underline{t}_c^*$ and $\underline{a}_c^*$ denote the dual spaces of the complexificaitons of the Lie algebras $\underline{t}$ and $\underline{a}$ of A respectively. Let $(\tau_\lambda, V_\lambda)$ $(\lambda \in \underline{t}_c^*)$ be the lowest K-type of π_Λ and let $(\sigma_\lambda, H_\lambda)$ be the representation of M given by restricting $\tau_\lambda(M)$ to the M-cyclic subspace H_λ generated by the highest weight vector of V_λ. Let

$$C^\infty(K, \sigma_\lambda) = \{f \in C^\infty(K, H_\lambda); f(mk) = \sigma_\lambda(m) f(k),\ m \in M, k \in K\}$$

$$(1.1)$$

$$C^\infty(G, \tau_\lambda) = \{f \in C^\infty(G, V_\lambda); f(kx) = \tau_\lambda(k) f(x),\ k \in K, x \in G\}.$$

Then the discrete series π_Λ is realized on the L^2 kernel of the Schmid operator D on $C^\infty(G, \tau_\lambda)$ (see [Sc] and §2 in [KW]) and the (non) unitary principal series $\pi_{\sigma_\lambda, \nu}$ $(\nu \in \underline{a}_c^*)$ is realized on $C^\infty(K,\sigma_\lambda)$ as the compact picture. According to the induced picture of $\pi_{\sigma_\lambda,\nu}$, each function $f \in C^\infty(K,\sigma_\lambda)$ can be extended to the function f on G by defining

$$f(ank)=e^{\nu(\log(a))}f(k) \quad (a \in A,\ n \in N \text{ and } k \in K) \qquad (1.2)$$

and this extension belongs to $C^\infty(G,\sigma_\lambda \times e^\nu)$. The Szegö map

$$S : C^\infty(K,\sigma_\lambda) \rightarrow C^\infty(G,\tau_\lambda) \qquad (1.3)$$

is defiend by

$$S(f)(x)=\int_K \tau_\lambda(k)^{-1} f(kx)dk. \qquad (1.4)$$

Knapp and Wallach in [KW] notice that the Szegö map S gives a relation between π_Λ and $\pi_{\sigma_\lambda,\nu}$; actually, for $\nu=\nu_\lambda \in \underline{a}_c^*$, defined by λ (see (6.5) in [KW]) S carries $C^\infty(K,\sigma_\lambda)$ into the kernel of the Schmid operator D on $C^\infty(G,\tau_\lambda)$ and then $\pi_{\sigma_\lambda,\nu_\lambda}$ onto π_Λ in an infinitesimally equivariant fashion. Here "infinitesimally" means that the correspondence holds between K-finite vectors of the domain and the range of the mapping. Therefore, as conjectured in §11 in [KW], it is worth realizing the discrete series π_Λ on the image of S without the K-finiteness assumption.

Now we assume that G has a simply connected complexification G_c and that G has real rank one. Then the above result can be extended to a singular integral form Λ such that $<\Lambda, \alpha_0>=0$ for a noncompact simple root α_0 and $<\Lambda, \beta>\neq 0$ for all other positive roots β. In this case we have two choices of the system of positive roots, say Δ^+ and $\Delta^{+\prime}=\Delta^+-\{\alpha_0\}\cup\{-\alpha_0\}$. Then we can define Szegö maps S and S' corresponding to Δ^+ and $\Delta^{+\prime}$ respectively (see [KW], §12). Since ν_λ equals ρ, the half the sum of the positive restricted roots with multiplicities, π_Λ corresponds to a limit of discrete sereis and $\pi_{\sigma_\lambda,\rho}$ to a reducible unitary principal series. Especially $\pi_{\sigma_\lambda,\rho}$ is infinitesimally equivalent with the direct sum of the K-finite images of S and S', which give two irreducible constituents of the reducible principal series (see [KW], Theorem 12.6).

The boundary value map

$$L : \text{the image of } S \to C^{\infty}(K, \sigma_{\lambda}) \tag{1.5}$$

is defined as follows.

$$L(S(f))(k)=\lim_{a\to\infty} E(e^{\rho(\log(a))}(\pi_{\sigma_{\lambda},\rho}(w^{-1}k)f)(a)) \quad (k\in K), \tag{1.6}$$

where E denotes the orthogonal projection from V_{λ} onto H_{λ} and w a representative of the nontrivial coset of the Weyl group W of A, which has order 2. Then in [B] Blank shows that in a G-equivariant fashion the composition map

$$L\circ S : C^{\infty}(K, \sigma_{\lambda}) \to C^{\infty}(K, \sigma_{\lambda}) \tag{1.7}$$

is a projection operator and, as shown in [KS], it consists of a linear combination of the identity operator and a principal value operator (see [B]). In his method the K-finiteness assumption does not required. This means that, in a global fashion, the limit of discrete series π_Λ ($\nu_\lambda=\rho$) is realized on the image of LoS equipped with the L^2-norm on K.

We retain the assumptions on G. Our aim is to give a global, not infinitesimal, realization of a discrete series. As mentioned above, when π_Λ is a limit of discrete series ($\nu_\lambda=\rho$), the Szegö map S : $C^\infty(K,\sigma_\lambda) \to C^\infty(G,\tau_\lambda)$ gives a global realization of π_Λ by taking the boundary value. Therefore, if we can shift the realization of the limit of discrete sereis π_Λ to a discrete sereis, we can construct the discrete series in a global fashion; therefore, the discrete series we shall treat below must be attached to a limit of discrete series. In order to shift the realization of π_Λ, we shall apply Zuckerman's technique introduced in [Z], roughly speaking, we shall form a suitable projection of tensor products of π_Λ and a finite dimensional representation of G.

Let μ be a dominant integral form on $\underline{t}$ and let (π,U) be a finite dimensional representation of G with lowest weight $-\mu$. Suppose that π satisfies a condition related with the order of weights, roughly speaking, this means that the Cayley transform carries $-\mu$ to the highest weight with respect to a noncompact

Cartan subalgebra. Then we can realize a discrete series, in a global fashion, as a subrepresentation of the nonunitary principal series $(\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}, C^\infty(K, \sigma_{\lambda-\mu}))$. Actually, first we take the tensor product of $\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}$ and π, then we define a map: $C^\infty(K, \sigma_{\lambda-\mu}) \to C^\infty(G, \sigma_\lambda \times e^\rho \times \beta)$, where β is the restriction of π to MAN; next we extract a component of $C^\infty(G, \sigma_\lambda \times e^\rho \times \beta)$, which is contained in $C^\infty(G, \sigma_\lambda \times e^\rho) = C^\infty(K, \sigma_\lambda)$, then we apply the Szegö maps S and S' on the component. Combining these proceedings, we can define nontrivial G-equivariant operators S_μ and S_μ': $C^\infty(K, \sigma_{\lambda-\mu}) \to C^\infty(G, H_\lambda)$. Let $\Omega_{\lambda,\mu}$ denote the kernel of S_μ' on $C^\infty(K, \sigma_{\lambda-\mu})$. Then $\Omega_{\lambda,\mu}$ is nontrivial, G-invariant and moreover, S_μ is injective on $\Omega_{\lambda,\mu}$. In their proofs we use the fact that the limit of discrete series π_Λ is realized in a global fashion. When G/K is hermitian, that is, G=SU(n,1), we see that $S_\mu(\Omega_{\lambda,\mu})$ is contained in $L^2(G, V_\lambda)$. Therefore, inducing the L^2 norm of $\Omega_{\lambda,\mu}$ from one of the image $S_\mu(\Omega_{\lambda,\mu})$, we can obtain a unitary representation $(\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$. Last we show that the representation is irreducible and matrix coefficients are square-integrable on G, so $(\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ is a discrete series of $G=SU(n,1)$.

This completes a global realization of a discrete series started with a limit of discrete series.

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