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Asymptotic behavior of Flensted-Jensen's spherical trace functions with respect to spectral parameters

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§0. Introduction

Let G be a connected real semisimple Lie group with finite center, σ be an involutive automorphism of G and H be an open subgroup of the fixed point group of σ. Then the homogeneous space G/H is called a semisimple symmetric space. Fix a σ-stable maximal compact subgroup K of G. Then under the assumption

$$\operatorname{rank}(G/H) = \operatorname{rank}(K/K \cap H), \tag{0.1}$$

Flensted-Jensen proves in **[FJ]** that there exist countably many discrete series for G/H, which are, by definition, equivalence classes of irreducible unitary representations of G realized in $L^2(G/H)$.

In **[FJ]** a certain function ψ_λ is introduced by the analytic continuation of the Poisson integral of a certain measure on a boundary of the non-compact riemannian form G^d/K^d of G/H, which is a K-finite simultaneous eigenfunction of the invariant differential operators on G/H. Here the parameter λ corresponds to the eigenvalue and also to the infinitesimal character of the representation generated by the function ψ_λ

on G/H. Then **[FJ]** proves that if λ is sufficiently regular, ψ_λ is square integrable on G/H under the condition (0.1) and thus we have discrete series for G/H.

In this paper we will proves an estimate for ψ_λ when λ tends to infinity. Denoting by C the normalizer of $K \cap H$ in K, Theorem 2.1 says that if x does not belong to the set CH/H, $\psi_\lambda(x)$ converges to 0 when λ tends to infinity under a certain condition. Here we remark that $|\psi_\lambda(x)| = 1$ if $x \in CH/H$.

Suppose Γ is a torsion free cocompact discrete subgroup of G such that $\Gamma \cap H$ is also cocompact subgroup of H. In **[T-W]** it is shown that the left translations of a certain cohomology class in $H^R(K\backslash G/\Gamma, E)$ corresponds to a discrete series for G/H. Here E is a locally constant bundle over $K\backslash G/\Gamma$ defined through a certain harmonic form on $K\backslash G/\Gamma$. In **[T-W]**, it is essential to consider the infinite sum

$$\sum_{\gamma\in\Gamma/\Gamma\cap H} \psi_\lambda(x\gamma), \tag{0.2}$$

which converges if $\psi_\lambda \in L^1(G/H)$. In fact the sum (0.2) converges when λ is sufficiently regular. On the other hand, the estimate obtained in this paper assures that the sum does not vanish for a sufficiently regular λ . This implies that the corresponding representation of G belonging to the discrete series for G/H is realized in $L^2(G/\Gamma)$. The estimate also gives a better condition for Γ in the main result in **[T-W, §6]**.

Suppose G is a compact semisimple Lie group. Then Flensted-Jensen's spherical trace function ψ_λ is nothing but a zonal spherical function on the compact symmetric space G/H. We first consider the estimate in this case.

§1. Zonal spherical functions

Let G_c be a simply connected and connected complex semisimple Lie group, θ be a holomorphic involutive automorphism of G_c and U be a θ-stable compact real form of G_c. We denote by $\mathfrak{g}_c$ and $\mathfrak{u}$ the Lie algebras of G_c and U, respectively. Let $\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ be the decomposition of $\mathfrak{u}$ into $+1$ and -1 eigenspaces for the induced involution, which will be also denoted by θ. Let define a real form $\mathfrak{g}$ of $\mathfrak{g}_c$ by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{t}$ be a maximal abelian subspace of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{j} = \mathfrak{t} + \mathfrak{a}_\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Σ and $\Sigma(\mathfrak{j})$ be the root systems for the pairs $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{g}_c, \mathfrak{j}_c)$, respectively. Here $\mathfrak{j}_c$ denotes the complexification of $\mathfrak{j}$ in $\mathfrak{g}_c$. We fix their compatible positive systems Σ^+ and $\Sigma(\mathfrak{j})^+$. Let G, K and $A_\mathfrak{p}$ be the analytic subgroups of G_c with the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{a}_\mathfrak{p}$, respectively, and $G = KA_\mathfrak{p}N$ be the Iwasawa decomposition of G corresponding to Σ^+. Let $\mathfrak{j}_c^*$ and $(\mathfrak{a}_\mathfrak{p})_c^*$ be the complexifications of the duals $\mathfrak{j}^*$ and $\mathfrak{a}_\mathfrak{p}^*$ of $\mathfrak{j}$ and $\mathfrak{a}_\mathfrak{p}$, respectively. We identify $\mathfrak{j}_c^*$ with $\mathfrak{j}_c$ by the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}_c$ and therefore we identify $(\mathfrak{a}_\mathfrak{p})_c^*$ and $\mathfrak{a}_\mathfrak{p}^*$ with subspaces of $\mathfrak{j}_c^*$.

Let $(U/K)^\frown$ be the set of equivalence classes of the irreducible unitary representations of U with non-trivial K-fixed vectors. Then $(U/K)^\frown$ is naturally identified with the set of equivalence classes of the finite dimensional irreducible holomorphic representations of G_c with non-trivial K-fixed vectors. Moreover owing to Cartan and Helgason [Wa, Theorem 3.3.1.1], the set of highest weights of the representations belonging to $(U/K)^\frown$ is given by

$$L = \left\{ \Lambda \in \mathfrak{a}_\mathfrak{p}^*; \frac{\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \{0, 1, 2, 3, ...\} \text{ for any } \alpha \in \Sigma^+ \right\}.$$

Let $\mathfrak{k}_c$ be the complexification of $\mathfrak{k}$ and K_c be the corresponding analytic subgroup of G_c. We denote by $\mathcal{O}(K_c \backslash G_c)$ the space of holomorphic functions on $K_c\backslash G_c$. Then $\mathcal{O}(K_c\backslash G_c)$ is a G_c-module thorough the right regular representation and it is known by a theorem due to Cartan that any element Λ in $(U/K)^\frown$ can be realized in a unique subspace V_Λ of $\mathcal{O}(K_c\backslash G_c)$. We denote by (π_Λ, V_Λ) the corresponding holomorphic representation of G_c in V_Λ and by f_Λ a highest weight vector of the representation with respect to the positive system $\Sigma(\mathrm{j})^+$. This implies

$$f_\Lambda(kg(\exp X)n) = f_\Lambda(g)\exp\Lambda(X) \tag{1.1}$$

for any $k \in K$, $g \in G_c$, $X \in \mathfrak{a}_\mathfrak{p}$ and $n \in N$. Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$ and f_Λ is holomorphic, we can normalize f_Λ by

$$f_\Lambda(e) = 1. \tag{1.2}$$

Let $N_U(K)$ be the normalizer of K in U. Note that $K\backslash N_U(K)$ is a finite group. Then we have the following

LEMMA 1.1.

$$|f_\Lambda(u)| \leq 1 \qquad \text{for any } u \in U \tag{1.3}$$

and

$$|f_\Lambda(u)| = 1 \qquad \text{for any } u \in N_U(K). \tag{1.4}$$

Moreover if Λ is regular (i.e. $\langle\Lambda, \alpha\rangle \neq 0$ for any $\alpha \in \Sigma$), then for any $u \in U - N_U(K)$ satisfying $|f_\Lambda(u)| = 1$, there exists an element $\widetilde{X} \in \mathfrak{k}$ with

$$\frac{\partial^2}{\partial t^2}|f_\Lambda(u\exp(t\widetilde{X}))|^2\Big|_{t=0} < 0. \tag{1.5}$$

PROOF: Fix a Hermitian inner product $(\ , \ )$ on V_Λ so that $\pi_\Lambda|_U$ is unitary and that it satisfies $(f_\Lambda, f_\Lambda) = 1$. Then $\pi_\Lambda(Z)$ is skew Hermitian for any $Z \in \mathfrak{u}$ and hence for $X \in \mathfrak{k}, Y \in \mathfrak{p}, u \in V_\Lambda$ and $v \in V_\Lambda$, we have $(u, \pi_\Lambda(X+Y)v) = (u, \pi_\Lambda(X)v - \sqrt{-1}\pi_\Lambda(\sqrt{-1}Y)v) = (\pi_\Lambda(-X+Y)u, v) = (\pi_\Lambda(-\theta(X+Y))u, v)$, which implies $(u, \pi_\Lambda(g)v) = (\pi_\Lambda(\theta(g^{-1}))u, v)$ for $g \in G$. Thus for $g \in G$, $f_\Lambda^2(g) = (\pi_\Lambda(g)f_\Lambda, \pi_\Lambda(g)f_\Lambda) = (\pi_\Lambda(\theta(g^{-1})g)f_\Lambda, f_\Lambda)$. Since f_Λ is a holomorphic function, we have

$$(1.6) \qquad f_\Lambda^2(g) = (\pi_\Lambda(\theta(g^{-1})g)f_\Lambda, f_\Lambda) \qquad \text{for any } g \in G_c.$$

This equation proves (1.3) because $\theta(u^{-1})u \in U$ for $u \in U$. Moreover (1.4) follows from the fact that the map $N_U(K) \ni u \mapsto f_\Lambda(u)$ induces a character of the finite group $K\backslash N_U(K)$.

Let u be an element in $U - N_U(K)$ with $|f_\Lambda(u)| = 1$. Then

$$(1.7) \qquad \pi_\Lambda(\theta(u^{-1})u)f_\Lambda = Cf_\Lambda$$

with a complex number C satisfying $|C| = 1$. Choose $Z \in \sqrt{-1}\mathfrak{a}_\mathfrak{p}$ so that $\exp Z \in KuK$. Then $\exp Z \notin N_U(K)$ and there exists an element X of the root space $\mathfrak{g}(\mathfrak{a}_\mathfrak{p}, \alpha)$ for a suitable root $\alpha \in \Sigma^+$ so that $\mathrm{Ad}(\exp Z)(X + \theta(X)) \notin \mathfrak{k}$. Let $\mathfrak{g}_0$ be the complex Lie subalgebra of $\mathfrak{g}_c$ generated by X and $\theta(X)$ and let G_0 be the corresponding analytic subgroup of G_c. Replacing X by cX with a suitable $c \in \mathbf{R}$ and putting $Y = -\theta(X)$ and $H = [X, Y]$, we may assume $[H, X] = 2X$ and $[H, Y] = -2Y$. Then $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$ with the correspondence

$$X \leftrightarrow X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y \leftrightarrow Y_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } H \leftrightarrow H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and G_0 is isomorphic to $SL(2, \mathrm{C})$ or $PSL(2, \mathrm{C})$. Putting $Z = Z_0 + Z_1$ with $Z_0 \in \sqrt{-1}\mathbf{R}H$ and $\alpha(Z_1) = 0$, we define

$$(1.8) \qquad \begin{aligned} h(t) &= f_\Lambda(\exp Z \exp t(X - Y)) \\ &= f_\Lambda(\exp Z_0 \exp t(X - Y)) f_\Lambda(\exp Z_1) \end{aligned}$$

for $t \in \mathbf{R}$.

Since the holomorphic function

$$h_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a^2 + c^2$$

on $SL(2, \mathrm{C})$ satisfies

$$h_0(\exp(s(X_0 - Y_0))g \exp tH_0 \exp rX_0) = h_0(g) \exp 2t$$

for any complex numbers s, t and r, we have

$$\begin{aligned} f_\Lambda(\exp Z_0 \exp t(X - Y)) &= h_0(\exp(\sqrt{-1}t_0 H_0) \exp t(X_0 - Y_0))^m \\ &= \left(e^{2\sqrt{-1}t_0} \cos^2 t + e^{-2\sqrt{-1}t_0} \sin^2 t\right)^m. \end{aligned}$$

Here $t_0 \in \mathbf{R}$ with $Z_0 = \sqrt{-1}t_0 H$ and m is a positive integer given by $m = \langle \Lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$. The assumption of the lemma implies that

$$\mathrm{Ad}(\exp \sqrt{-1}t_0 H_0)(X_0 - Y_0) \notin \mathrm{C}(X_0 - Y_0)$$

and therefore $e^{\sqrt{-1}t_0} \neq \pm 1, \pm\sqrt{-1}$. This assures that there exists $t \in \mathbf{R}$ with $|h(t)| = |f_\Lambda(\exp Z_0 \exp t(X - Y))| < 1$.

Thus we have proved the existence of $\widetilde{X} \in \mathfrak{k}$ satisfying $|f_\Lambda(u \exp \widetilde{X})| < 1$. Denoting $u' = \theta(u^{-1})u$, we put

$$\pi_\Lambda(\mathrm{Ad}(u')\widetilde{X} - \widetilde{X})f_\Lambda = Cf_\Lambda + v$$

with suitable $C \in \mathbf{C}$ and $v \in V_\Lambda$ satisfying $(f_\Lambda, v) = 0$.

Suppose $v = 0$. Since Λ is regular, we have $\mathrm{Ad}(u')\widetilde{X} - \widetilde{X} \in \sqrt{-1}\mathfrak{a}_\mathfrak{p}$ and

$$\begin{aligned}\langle \mathrm{Ad}(u')\widetilde{X} - \widetilde{X}, \mathrm{Ad}(u')\widetilde{X} - \widetilde{X}\rangle &= \langle \mathrm{Ad}(u')\widetilde{X}, \mathrm{Ad}(u')\widetilde{X}\rangle - \langle \widetilde{X}, \widetilde{X}\rangle \\ &= 0.\end{aligned}$$

Hence $Ad(u')\widetilde{X} = \widetilde{X}$ and therefore $\theta(u^{-1})u = \theta((u \exp \widetilde{X})^{-1})u \exp \widetilde{X}$. Combining this with (1.6) and (1.7), we have a contradiction.

Thus we can conclude that $v \neq 0$. For $t \in \mathbf{R}$, put

$$\pi_\Lambda(\exp(-t\widetilde{X})\exp(tAd(u')\widetilde{X}))f_\Lambda = c(t)f_\Lambda + tv(t).$$

Here $c(t) \in \mathbf{C}$ and $v(t) \in V_\Lambda$ with $(v(t), f_\Lambda) = 0$ and $v(0) = v$. Since $|f_\Lambda(u' \exp t\widetilde{X})|^4 = |c(t)|^2$ and since $|c(t)|^2 + t^2(v(t), v(t)) = 1$, we have the last claim in the lemma. ∎

The zonal spherical functions ϕ_Λ on the simply connected compact symmetric space U/K are parametrized by the elements Λ of L and are given by

(1.9) $$\phi_\Lambda(g) = \int_K f_\Lambda(gk)dk$$

with the normalized Haar measure dk on K. They have the following asymptotic for Λ:

PROPOSITION 1.2.

$$|\phi_\Lambda(u)| = 1 \qquad \text{for any } u \in N_U(K). \tag{1.10}$$

Conversely for any compact subset V of $U - N_U(K)$, there exists a positive number C_V such that

$$|\phi_\Lambda(u)| \leq \left(1 + C_V \min_{\alpha\in\Sigma^+} \langle \Lambda, \alpha\rangle^{\frac{1}{2}}\right)^{-1} \qquad \text{for any } u \in V. \tag{1.11}$$

PROOF: Since K is a normal subgroup of $N_U(K)$, $f_\Lambda(gk) = f_\Lambda(g)$ for any $g \in N_U(K)$ and $k \in K$ and therefore we have (1.10) from (1.9) and Lemma 1.1.

Fix a regular $\Lambda_0 \in L$ and put $m(\Lambda) = \min_{\alpha\in\Sigma^+}\langle\Lambda,\alpha\rangle$. Since

$$f_{\Lambda+\Lambda'} = f_\Lambda f_{\Lambda'} \qquad \text{for } \Lambda, \Lambda' \in L,$$

we have a positive constant C with $|f_\Lambda(g)|^\Lambda \leq |f_{\Lambda_0}(g)|^{Cm(\Lambda)}$. Here we remark the existence of a positive number C' which satisfies $m(\Lambda) \geq C'$ for any regular $\Lambda \in L$. Then dividing K into a finite number of sufficiently small disjoint open subsets V_i with $\int_{K-\bigcup V_i} dk = 0$ and applying the following lemma to the function $|f_{\Lambda_0}(g)|^2$, we have the proposition by Lemma 1.1. ∎

LEMMA 1.3. *Let $f(x)$ be a real valued C^∞-function defined on a neighborhood of the origin of $\mathbf{R}^n$ which satisfies $0 \leq f(x) \leq 1$. Suppose*

$f(0) < 1$ or $\frac{\partial^2 f}{\partial x_1^2}(0) < 0$. Then for any sufficiently small connected open neighborhood V of the origin, there exists a positive number ε such that

$$\int_V f(x)^\lambda dx \le \left(1 + \varepsilon \frac{\lambda}{1+\lambda^{\frac{1}{2}}}\right)^{-1} \int_V dx$$

for any positive real number λ.

PROOF: We may assume $f(0) = 1$ because the lemma is clear if $f(0) < 1$. Denoting $x' = (x_2, \ldots, x_n)$, there exist C^∞-functions $a(x)$, $b(x')$ and $c(x')$ such that

$$f(x)^{-1} = 1 + a(x)(x_1^2 + 2b(x')x_1 + c(x')).$$

in a neighborhood of the origin. Here the assumption implies $a(0) > 0$ and $c(x') \ge 0$. Replacing x_1 by $x_1 - b(x')$, we may moreover assume $b(x') = 0$. For a small connected open neighborhood V of the origin, we can choose a positive number C which satisfies $a(x) \ge C$ if $x \in V$.

Let $v(x_1)$ be the volume of the open set $\{x' \in \mathbf{R}^{n-1}; (x_1, x') \in V\}$ in $\mathbf{R}^{n-1}$. Put $t(x_1) = \int_0^{x_1} v(x_1)dx_1, t_+ = t(\infty)$ and $t_- = t(-\infty)$. Then $t_+ > 0$, $t_- < 0$, $\int_V dx = t_+ - t_-$ and there exists a positive number C' with $Cx_1^2 \ge C't(x_1)^2$. Thus we have

$$\begin{aligned} \int_V f(x)^\lambda &\le \int_V (1 + Cx_1^2)^{-\lambda} dx \\ &= \int v(x_1)(1 + Cx_1^2)^{-\lambda} dx_1 \\ &\le \int_{t_-}^{t_+} (1 + C'\lambda t^2)^{-1} dt \\ &= \frac{1}{\sqrt{C'\lambda}} (\arctan \sqrt{C'\lambda} t_+ - \arctan \sqrt{C'\lambda} t_-). \end{aligned}$$

Since $s^{-1}\arctan s \leq \left(1+C''\frac{s^2}{1+s}\right)^{-1}$ for all $s>0$ with a suitable $C''>0$, the lemma is clear. ∎

§2. Flensted-Jensen's spherical trace functions

Retain the notation in §1. Let σ be an involutive automorphism of G which commutes with θ, $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of the induced involution denoted also by σ and H be the analytic subgroup of G with the Lie algebra $\mathfrak{h}$. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}\cap\mathfrak{q}$. In this section we assume

$$\text{(2.1)}\qquad \operatorname{rank}(G/K)=\operatorname{rank}(H/H\cap K),$$

which assures that we can choose $\mathfrak{a}_\mathfrak{p}\subset\mathfrak{h}\cap\mathfrak{p}$. For an element $\lambda\in(\mathfrak{a}_\mathfrak{p})^*_c$ Flensted-Jensen [FJ] defined a function

$$\text{(2.2)}\qquad \psi_\lambda(x)=\int_{K\cap H} e^{\langle-\lambda-\rho,H(x)\rangle}dk$$

on G. Here dk is the normalized Haar measure on $K\cap H$, ρ is half the sum of roots in Σ^+ counted with multiplicity and the element $H(x)$ in $\mathfrak{a}_\mathfrak{p}$ is defined by the Iwasawa projection with $\exp H(x)\in KxN$. Let $\Sigma(\mathfrak{h},\mathfrak{a}_\mathfrak{p})$ be the root system for the pair $(\mathfrak{h},\mathfrak{a}_\mathfrak{p})$, $\Sigma(\mathfrak{h},\mathfrak{a}_\mathfrak{p})^+$ be its positive system with $\Sigma(\mathfrak{h},\mathfrak{a}_\mathfrak{p})^+\subset\Sigma^+$ and ρ_t be half the sum of roots in $\Sigma(\mathfrak{h},\mathfrak{a}_\mathfrak{p})^+$ counted with multiplicity in $\mathfrak{h}$. Then [FJ, (3.13)] says

$$\text{(2.3)}\qquad \psi_\lambda(\ell\exp(X))=\int_{K\cap H} e^{\langle-\lambda-\rho,H(\exp(-X)k)\rangle}e^{\langle\lambda+\rho-2\rho_t,H(\ell k)\rangle}dk$$

for $\ell\in H$ and $X\in\mathfrak{p}\cap\mathfrak{q}$.

Put $\mathfrak{k}^d = \mathfrak{k} \cap \mathfrak{h} + \sqrt{-1}(\mathfrak{p} \cap \mathfrak{h})$, $\mathfrak{h}^d = \mathfrak{k} \cap \mathfrak{h} + \sqrt{-1}(\mathfrak{k} \cap \mathfrak{q})$ and $\mathfrak{g}^d = \mathfrak{k}^d + \sqrt{-1}(\mathfrak{k} \cap \mathfrak{q}) + \mathfrak{p} \cap \mathfrak{q}$. Let G^d be the simply connected Lie group with the Lie algebra $\mathfrak{g}^d$ and let K^d and H^d be the analytic subgroups of G^d with the Lie algebras $\mathfrak{k}^d$ and $\mathfrak{h}^d$, respectively. Then K^d is a maximal compact subgroup of G^d modulo center of G^d. Denoting

(2.4) $\quad L' = \{\Lambda \in \mathfrak{a}_{\mathfrak{p}}^*; \Lambda$ coincides with the highest weight of an irreducible unitary representation of K^d with a non-trivial $K^d \cap H^d$-fixed vector$\}$

under the ordering defined by $\Sigma(\mathfrak{h}, \mathfrak{a}_{\mathfrak{p}})^+$, we assume that $\lambda \in (\mathfrak{a}_{\mathfrak{p}})_c^*$ satisfies

$$\langle \lambda + \rho, \alpha \rangle \geq 0 \qquad \text{for } \alpha \in \Sigma^+ \tag{2.5}$$

and

$$\lambda + \rho - 2\rho_t \in L'. \tag{2.6}$$

The condition (2.6) assures that the function

$$\mathfrak{h} \ni Y \mapsto e^{\langle \lambda + \rho - 2\rho_t, H(\exp(Y)) \rangle}$$

can be holomorphically extended on the complexification of $\mathfrak{h}$ and defines a function on K^d. Since $G^d = K^d \exp(\mathfrak{a}) H^d$, we define a function $\psi_\lambda(gH^d)$ on the semisimple symmetric space G^d/H^d by the right hand of (2.3) with $\ell \in K^d$ and $X \in \mathfrak{a}$.

Then **[FJ]** proves that $\psi_\lambda(gH^d)$ is a simultaneous eigenfunction of the invariant differential operators on G^d/H^d and moreover that it is square integrable modulo the center of G^d with respect to the invariant measure on G^d/H^d if λ is sufficiently regular. Moreover [MO] proves that the condition $\langle\lambda,\alpha\rangle > 0$ for $\alpha \in \Sigma^+$ with (2.6) is equivalent to the square integrability. To have an estimate of this function when $\langle\lambda,\alpha\rangle$ tends to infinity for all $\alpha \in \Sigma^+$, we review the argument in [FJ].

Let $\{\alpha_1,\ldots,\alpha_n\}$ be the fundamental system for Σ^+. Then it follows from [FJ, Lemma 4.6 and Lemma 4.7] that there exist finite number of real analytic functions $b_{ij}(k)$ on $K \cap H$ and elements μ_{ij} of $\mathfrak{a}_\mathfrak{p}^*$ ($i = 1,\ldots,d_i$ and $j = 1,\ldots,n$) such that for $X \in \mathfrak{a}$ and $k \in K \cap H$

$$(2.7)\qquad e^{\langle\lambda+\rho,H(\exp(X)k)\rangle} = \prod_{j=1}^{n}\Big(\sum_{i=1}^{d_j}|b_{ij}(k)|^2\cosh\langle\mu_{ij},X\rangle\Big)^{\frac{\langle\lambda+\rho,\alpha_j\rangle}{2\langle\alpha_j,\alpha_j\rangle}},$$

$$\sum_{i=1}^{d_j}|b_{ij}(k)|^2 = 1 \qquad \text{for } j = 1,\ldots,n$$

and

$$(2.8)\ e^{\langle\lambda+\rho,H(\exp(X)k)\rangle} > 1 \qquad \text{if } X \neq 0 \text{ and } \langle\lambda+\rho,\alpha\rangle > 0 \text{ for } \alpha \in \Sigma^+.$$

Since $K \cap H$ is compact, we can conclude the existence of a positive number ε satisfying

$$(2.9)\qquad e^{\langle-\lambda-\rho,H(\exp(-X)k)\rangle} \leq \Big(\cosh(\varepsilon\min_{\alpha\in\Sigma^+}\langle\lambda+\rho,\alpha\rangle\|X\|)\Big)^{-1}$$

for $X \in \mathfrak{a}$ and $k \in K \cap H$ by denoting $\|X\| = \langle X,X\rangle^{1/2}$.

Let $\mathfrak{k}_0^d$ be the center of $\mathfrak{k}^d$ and put $\mathfrak{k}_r^d = [\mathfrak{k}^d,\mathfrak{k}^d]$. Let K_0^d and K_r^d be the analytic subgroups of K^d with the Lie algebras $\mathfrak{k}_0^r$ and $\mathfrak{k}_0^d$, respectively.

Then $K^d = K_0^d K_r^d$ and for $k_0 \in K_0^d$ and $k_r \in K_r^d$

$$e^{\langle \lambda+\rho-2\rho_t, H(k_0 k_r)\rangle} = e^{\langle \lambda+\rho-2\rho_t, H(k_0)\rangle} e^{\langle \lambda+\rho-2\rho_t, H(k_r)\rangle}.$$

Since $|e^{\langle \lambda+\rho-2\rho_t, H(k_0)\rangle}| = 1$, by applying the argument of the proof of Proposition 1.2 to the function

$$K_r^d \ni k_r \mapsto e^{\langle \lambda+\rho-2\rho_t, H(k_r)\rangle}$$

we have the following

THEOREM 2.1. *Under the above notation we can find a positive number ε which implies the following:*

Let λ be an element satisfying (2.5) and (2.6). Then

$$|\psi_\lambda(k \exp(X) H^d)| \leq \Big(\cosh\big(\varepsilon \|X\| \min_{\alpha \in \Sigma+} \langle \lambda + \rho, \alpha\rangle\big)\Big)^{-1}$$

for $k \in K^d$ and $X \in \mathfrak{a}$.

Let A be the normalizer of $K^d \cap H^d$ in K^d. Then for any compact subset V of $K^d - A$ there exists a positive number C_V such that

(2.10)
$$|\psi_\lambda(k_0 k \exp(X) H^d)| \leq \Big(\cosh\big(\varepsilon \|X\| \min_{\alpha \in \Sigma+} \langle \lambda + \rho, \alpha\rangle\big)\Big)^{-1} \cdot \Big(1 + C_V \min_{\alpha \in \Sigma(\mathfrak{h}, \mathfrak{a}_\mathfrak{p})+} \langle \lambda + \rho - 2\rho_t, \alpha\rangle^{1/2}\Big)^{-1}$$

for $k_0 \in K_0^d$, $k \in V$ and $X \in \mathfrak{a}$.

We will give an application of the above theorem. Fix a subgroup Z of G^d contained in the center of G^d so that K^d/Z is compact and put $G' = G^d/Z, K' = K^d/Z$ and $H' = H^dZ/Z$. Suppose $\lambda + \rho - 2\rho_t$ equals the highest weight of a certain irreducible unitary representation of K' with a non-trivial $K' \cap H'$-fixed vector. Then the function ψ_λ defines a square integrable function on G'/H' if $\langle \lambda, \alpha \rangle > 0$ for $\alpha \in \Sigma^+$.

Let Γ be a discrete subgroup of G' so that

$$H'/\Gamma \cap H' \ \text{ has a finite volume.} \tag{2.11}$$

We denote by $Z(\mathfrak{g})$ the center of the universal enveloping algebra of $\mathfrak{g}_c$ and by $N_{K'}(K' \cap H')$ the normalizer of $K' \cap H'$ in K'. Then [T-W, Lemma 3.7] says that if a function $\psi \in L^1(G'/H')$ is $Z(\mathfrak{g})$-finite and left K'-finite, the series

$$\sum_{\gamma \in \Gamma/\Gamma \cap H'} \psi(g\gamma) \tag{2.12}$$

converges absolutely and uniformly on compact subsets of G' and defines a function on G'/Γ.

Applying this to the function ψ_λ, we can conclude the existence of a positive number M such that the series

$$\phi_\lambda(g) = \sum_{\gamma \in \Gamma/\Gamma \cap H'} \psi_\lambda(g\gamma) \tag{2.13}$$

converges absolutely and uniformly on every compact subset of G' and defines a non-zero function ϕ_λ on G'/Γ if

$$\Gamma \cap N_{K'}(K' \cap H') = \{e\} \tag{2.14}$$

and

$$(2.15) \qquad \langle \lambda, \alpha \rangle \geq M \qquad \text{for } \alpha \in \Sigma^+.$$

In fact, if λ satisfies (2.15) for a sufficiently large M, then it follows from Theorem 2.1 that $\phi_\lambda \in L^1(G'/H')$ (cf. [**O**, Corollary 1.3] or [**T-W**, Proposition 2.4]) and that $\phi_\lambda(e) \neq 0$. Combining this fact with the main result in [**M-O**], we have

THEOREM 2.2. *Let G be a connected real semisimple Lie group with finite center, σ be an involutive automorphism of G and H be an open subgroup of the fixed point group of σ. Assume (0.1) in place of (2.1). Let Γ be a torsion free discontinuous subgroup of G so that G/Γ is compact and that $H/\Gamma \cap H$ has finite volume. Then every discrete series for G/H with a sufficiently regular infinitesimal character can be realized in $L^2(G/\Gamma)$.*

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