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Invariant system of differntial equations on Siegel's upper half-plane

by

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§1. Introduction

The present paper deals with the Poisson integral representation of solutions of a certain system of differential equations on Siegel's upper half plane $\mathfrak{H}_n$;

$$\mathfrak{H}_n = \{Z \in M_n(\mathbb{C});\ {}^tZ = Z,\ \mathrm{Im}\, Z > 0\}.$$

Let us introduce the differential operator

$$\partial_Z = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}}\right),$$

and consider the system of differential equations:

$$\mathcal{M}_s : \begin{cases} Du = \gamma(D)(\lambda_s)u \\ Y\,{}^t(Y\partial_Z)\bar{\partial}_Z u = \frac{1}{16}(s+\frac{n+1}{2})(s-\frac{n+1}{2})u \\ Y\,{}^t(Y\bar{\partial}_Z)\partial_Z u = \frac{1}{16}(s+\frac{n+1}{2})(s-\frac{n+1}{2})u \end{cases}$$

(cf. §5). We denote by $\mathcal{A}(\mathfrak{H}_n, \mathcal{M}_s)$ the space of real analytic functions on $\mathfrak{H}_n$ which are solutions of $\mathcal{M}_s$. Put

$$P_s(Z) = (\det Y)^{\frac{1}{2}(s+\frac{n+1}{2})} |\det Z|^{-(s+\frac{n+1}{2})}.$$

Let $\mathcal{B}(S_n)$ be the space of hyperfunctions on the Bergman-Šilov boundary S_n of $\mathfrak{H}_n$. For any $f \in \mathcal{B}(S_n)$, we put

$$(\mathcal{P}_s f)(Z) = \int_{S_n} f(X')P_s(X'-X,Y)dX'.$$

Our main result is, then, stated as follows.

Assume $s \in \mathbb{C}$ satisfies the condition

$$n - 2s - 2 \notin \mathbb{N}_+.$$

Then any solution u of the system $\mathfrak{M}_s$ can be represented as

$$u = \mathcal{P}_s f$$

for some hyperfunction f on S_n.

This paper is organized as follows. In section 2, after introducing notation concerning semisimple Lie groups, we state the main result in [6] which will be used in the proof of our theorem. From §3 to the last we assume that X is Siegel's upper half plane. In section 3, we will realize X in two ways, one which is the usual realization as Siegel's upper half plane and another in the alosure of which the Martin boundary appears. We will define the Poisson integral $\mathcal{P}_s f$ for any hyperfunction f on S_n in §4. Section 5 is devoted to the proof of the main result.

The Poisson transform $\mathcal{P}_s$ is a G-isomorphism of $\mathcal{B}(S_n)$ onto the solution space of the space $\mathfrak{M}_s$.

§2 Notation and preliminary results.

We shall use the standard notation $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the ring of integers, the field of real numbers and the field of complex numbers, respectively. The set of non-negative, positive integers and positive real numbers are denoted by $\mathbb{N}$, $\mathbb{N}_+$ and $\mathbb{R}_+$, respectively. For any real analytic manifold X, we denote by $\mathcal{A}(X)$ (resp. $\mathcal{B}(X)$) the space of all real analytic functions (resp. hyperfunctions) on X.

Let X be a symmetric space of the noncompact type, that is, a coset space $X = G/K$ where G is a noncompact, connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $G = KAN$ be an Iwasawa decomposition of G, the groups A and N being Abelian and nilpotent, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the derect Lie algebra decomposition which corresponds to the decomposition $G = KAN$. If $g \in G$, let $H(g) \in \mathfrak{a}$, $k(g) \in K$, $n(g) \in N$ be determined by $g = k(g)(\exp H(g))n(g)$. Let $\mathfrak{a}^*$ (respectively $\mathfrak{a}_c^*$) denote the space of real-valued (respectively complex-valued) linear forms on $\mathfrak{a}$ and for each $H \in \mathfrak{a}$ let $\rho(H)$ denote half the trace of the linear transformation $X \to [H,X]$ of $\mathfrak{n}$. If $\lambda \in \mathfrak{a}_c^*$ the function

$$\varphi_\lambda(gK) = \int_K \exp\{(\lambda - \rho)(H(gk))\}dk$$

(dk is Haar measure) is a spherical function. Let $\mathbb{D}(X)$ denote the algebra of invariant differential operators on X. Then $\varphi_\lambda(gK)$ is an eigenfunction for any $D \in \mathbb{D}(X)$. Let, for any $D \in \mathbb{D}(X)$, $\gamma(D)(\lambda)$ denote the eigenvalue of D, that is,

$$D\varphi_\lambda = \gamma(D)(\lambda)\varphi_\lambda .$$

Fix λ in $\mathfrak{a}_c^*$. We denote by $\mathcal{A}(X;\mathcal{M}(\lambda))$ the space of all real analytic functions on X satisfying the system $\mathcal{M}(\lambda)$ of differential equations;

$$\mathcal{M}(\lambda) : Du = \gamma(D)(\lambda)u \qquad (D \in \mathbb{D}(X)).$$

Let M be the centralizer of A in K and put $P = MAN$. Let $L(\lambda)$ be the line bundle on G/P associated with the character of $P = MAN$; $man \to e^{(\rho-\lambda)(\log a)}$. We denote by $\mathcal{B}(G/P; L(\lambda))$ the space of all $L(\lambda)$-valued hyperfunctions on G/P. Then $\mathcal{B}(G/P; L(\lambda))$ is canonically identified with the space of all hyperfunctions f on G such that

$$f(gman) = e^{(\lambda-\rho)(\log a)}f(g) \quad (g \in G,\ m \in M,\ a \in A \text{ and } n \in N).$$

For any g in G and f in $\mathcal{B}(G/P;L(\lambda))$, we put $(\pi_\lambda(g)f)(x) = f(g^{-1}x)$ $(x \in G)$, then it is obvious that $\pi_\lambda(g)f$ is again an element of $\mathcal{B}(g/P; L(\lambda))$ and π_λ defines a representation of G on $\mathcal{B}(G/P; L(\lambda))$. Also, for any g in G and u in $\mathcal{A}(X; \mathcal{M}(\lambda))$ we put $(\pi(g)u)(x) = u(g^{-1}x)$ $(x \in G)$. Then it is clear that $\mathcal{A}(X; \mathcal{M}(\lambda))$ is stable under $\pi(g)$ and we obtain a representation of G on $\mathcal{A}(X; \mathcal{M}(\lambda))$.

For any element f of $\mathcal{B}(G/P; L(\lambda))$, we define the Poisson transform $\mathcal{P}_\lambda f$ on X by

$$(\mathcal{P}_\lambda f)(gK) = \int_K f(gk)dk.$$

By a simple calculation it follows that

$$(\mathcal{P}_\lambda f)(gK) = \int_K f(k)(\pi(k)\Psi_\lambda)(gK)dk,$$

where

$$\Psi_\lambda(gK) = \exp\{-(\lambda+\rho)(H(g^{-1}))\}.$$

It is widely known that, for any fixed k in K, $\pi(k)\Psi_\lambda$ belongs to $\mathcal{A}(X;\mathcal{M}(\lambda))$. Hence the image of $\mathcal{P}_\lambda$ is contained in $\mathcal{A}(X;\mathcal{M}(\lambda))$.

The main result in [G] gives the characterization of the image of $\mathcal{P}_\lambda$. Namely

Theorem 1. We assume the condition

$$\text{(A)}' \qquad -2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \notin \mathbb{N}_+ \qquad \text{for any } \alpha \in \Sigma^+.$$

Then the Poisson integral $\mathcal{P}_\lambda$ is a G-isomorphism of $\mathcal{B}(G/P;\ L(\lambda))$ onto $\mathcal{A}(X;\mathcal{M}(\lambda))$.

§3. Two realizations of $Sp(n,\mathbb{R})/U(n)$.

From now on, we deal with the Lie group $G = Sp(n,R)$. The associated symmetric space $X = G/K$ is identified with Siegel's upper half plane $\mathfrak{H}_n$ of degree n;

$$\mathfrak{H}_n = \{Z = X + \sqrt{-1}Y;\ X \text{ and } Y \text{ are } n\times n \text{ real matrices and } Y \text{ is positive-definite.}\}.$$

We shall first give the concrete correspondence of X with $\mathfrak{H}_n$ and next define another realization of X in the closure of which the Martin boundary appears.

We now introduce some notation for later convenience. Let $g = sp(n,\mathbb{R})$ be the Lie algebra of G. Let E_{ij} be the matrix of order n whose (i,j)-entry is 1 and others are zero and put

$$E_{ij}^0 = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \qquad E_{ij}^- = \begin{bmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{bmatrix}.$$

Then E_{ij}^0, E_{ij}^- and E_{ij}^+ $(= {}^tE_{ij}^-)$ constitute a basis of g. We take

$$k = \sum_{i=j} \mathbb{R}(E_{ij}^0 - E_{ji}^0) + \sum_{i,j} \mathbb{R}(E_{ij}^+ - E_{ij}^-)$$

as a maximal compact subalgebra of g and take

$$a = \sum_{i=1}^{n} \mathbb{R}E_i^0$$

as a maximal abelian subalgebra of g. We denote by e_i the linear form on a defined by

$$e_i(E_j^0) = \delta_{ij}.$$

Then $e_1,\ldots,e_n$ form a dual basis of a^*. The root system Σ of (g,a) is

$$\Sigma = \{\pm(e_i \pm e_j)(i\neq j),\ \pm 2e_i\}.$$

We take $a^+ = \{ \sum_{i=1}^{n} t_i E_i^0 \ ;\ t_1>\cdots>t_n\}$ as the positive Weyl chamber of a. Then

$$\Psi = \{\ \alpha_i = e_i - e_{i+1}\ (i = 1,\ldots,n-1),\ \alpha_n = 2e_n\},$$

$$\Sigma^+ = \{\ e_i - e_j(i < j),\ e_i + e_j(i,j \text{ are arbitrary})\}$$

are the set of positive simple roots and that of positive roots respectively. In this ordering on Σ, we have

$$n = \sum_{i<j} \mathbb{R}E_{ij}^0 + \sum_{i,j} \mathbb{R}E_{ij}^+,$$

$$\bar{n} = \sum_{i>j} \mathbb{R}E_{ij}^0 + \sum_{i,j} \mathbb{R}E_{ij}^-.$$

We decompose a and $\bar{n}$ into two parts as follows:

1) $a = a_1 + a_2$,

where

$$a_1 = \sum \mathbb{R}(E_i^0 - E_{i+1}^0),$$

$$a_2 = \mathbb{R}H_0 \quad (H_0 = \sum_{i=1}^{n} E_i^0).$$

2) $\bar{n} = \bar{n}_1 + \bar{n}_2$,

where

$$\bar{n}_1 = \sum_{i>j} \mathbb{R}E_{ij}^0$$

$$\bar{n}_2 = \sum_{i,j} \mathbb{R}E_{ij}^-.$$

We denote by A_1, A_2, $\bar{N}_1$ and $\bar{N}_2$ the analytic subgroups of G corresponding to a_1, a_2, $\bar{n}_1$ and $\bar{n}_2$ respectively. Furthermore we put

$$M_1 = \left\{ \begin{bmatrix} g & \\ & {}^t g^{-1} \end{bmatrix} ;\ g \in SL(n,\mathbb{R}) \right\}.$$

Then M_1 centralize A_2. We put $P = MAN$ and $P_1 = M_1A_1N$. Then P and P_1 are parabolic subgroups of g.

We give the correspondence of X with $\mathfrak{h}_n$. We denote by $S(n,\mathbb{R})$ (resp. $S^+(n,\mathbb{R})$) the set of real symmetric matrices of order n (resp. that of positive-definite ones in $S(n,\mathbb{R})$). For any $X \in S(n,\mathbb{R})$, put

$$\bar{n}_1(X) = \begin{vmatrix} 1 & \\ X & 1 \end{vmatrix}$$

and, for any $Y \in S^+(n,\mathbb{R})$, put

$$a(Y) = \begin{bmatrix} Y^{-\frac{1}{2}} & \\ & Y^{\frac{1}{2}} \end{bmatrix}.$$

They are clearly contained in G. Furthermore, for any $g \in G$, there exist unique elements $X \in S(n,\mathbb{R})$, $Y \in S^{+}(n,\mathbb{R})$ and $k \in K$ such that $g = \bar{n}_1(X)a(Y)k$. This follows from the Iwasawa decomposition $G = \bar{N}AK$. Noting this fact, the correspondence

$$G/K \ni \bar{n}_1(X)a(Y)K \longrightarrow X + \sqrt{-1}Y \in \mathfrak{H}_n$$

defines an analytic isomorphism of G/K onto $\mathfrak{H}_n$. If we put

$$g\bar{n}_1(X)a(Y)K = \bar{n}_1(X')a(Y')K$$

for $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$, the straightforward calculation shows that

$$Z' = (C + DZ)(A + BZ)^{-1},$$

where $Z = X + \sqrt{-1}Y$ and $Z' = X' + \sqrt{-1}Y'$. Hence, under the above identification, each $g \in G$ gives rise to a holomorphic diffeomorphism

$$T_g : Z \longmapsto (C + DZ)(A + BZ)^{-1}$$

of $\mathfrak{H}_n$, the matrix g being written as above. Since $\mathfrak{H}_n$ is imbedded in $\check{S}(n,\mathbb{C})$ which is the set of all complex symmetric matrix of order n, the Bergman-Šilov boundary*) of $\mathfrak{H}_n$ is realized as the set $S(n,\mathbb{R})$ under the natural inclusion into $\check{S}(n,\mathbb{C})$. More precisely, the B.-S. boundary of G/K is G/P_1 and $S(n,\mathbb{R})$ is naturally imbedded in G/P_1 as an open dense subset under the correspondence

*) We abbreviate the B.-S. boundary in the sequel.

$$S(n,\mathbb{R}) \ni X \longrightarrow \bar{n}_1(X)P_1 \in G/P_1.$$

This follows from the fact that $\bar{N}_1 P_1$ is an open dense subset of G.

Next we realize G/K by the method due to T.Oshima [10] (cf. also Satake [13], Furstenberg [1]). We identify A with $\mathbb{R}_+^n$ by the correspondence

$$A \ni a \longrightarrow (e^{-\alpha_1(\log a)}, \ldots, e^{-\alpha_n(\log a)}) \in \mathbb{R}_+^n.$$

For any $t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$, we define $a(t) \in A$ so that t corresponds to $a(t)$ under this map, namely,

$$t_i = \exp\{-\alpha_i(\log a(t))\} \quad (i = 1, \ldots, n).$$

Let $\hat{X}_+ = G \times \mathbb{R}_+^n$ and $\hat{X} = G \times \mathbb{R}^n$. Then G acts on $\hat{X}$ and $\hat{X}_+$ by the correspondence $(g', (g,t)) \to (g'g, t)$ for $g, g' \in G$, $t \in \mathbb{R}^n$ (or $t \in \mathbb{R}_+^n$). We define an equivalence relation $\underset{+}{\sim}$ for points in X_+. Two points $x = (g,t)$ and $x' = (g',t')$ in X_+ are equivalent, which will be denoted by $x \underset{+}{\sim} x'$, if and only if $ga(t)K = g'a(t')K$. Then it follows from the same argument as in the proof of Lemma 7 in [10] that this equivalence relation $\underset{+}{\sim}$ naturally induces an equivalence relation $\sim$ for points in $\hat{X}$. We put $\tilde{X} = \hat{X}/\sim$ and let π be the natural projection of $\hat{X}$ onto $\tilde{X}$. Since the action of G on $\hat{X}$ is compatible with the equivalence relation, G also acts on $\tilde{X}$. Put $\tilde{U}_g = \pi(g\bar{N} \times \mathbb{R}^n)$ for $g \in G$. Then the map $(\bar{n}, t) \to (g\bar{n}, t)$ defines the analytic diffeomorphism

$$\varphi_g : \bar{N} \times \mathbb{R}^n \longrightarrow \tilde{U}_g.$$

Then $\tilde{X}$ has the properties mentioned in Theorem 5 in [10]. In particular, $\tilde{U}_g$ is an open dense submanifold of $\tilde{X}$ and $U_g^+ = \pi(g\bar{N} \times \mathbb{R}_+^n)$ (resp. $U_g^0 = \pi(g\bar{N} \times \{0\})$) is isomorphic to G/K (resp. an open dense submanifold of G/P). Furthermore, $\tilde{X} = \bigcup_{g\in G} \tilde{U}_g$. Hence $\{U_g^0 ; g\in G\}$ is an open covering of G/P, which is the Martin boundary (we denote the M. boundary in the sequel).

We briefly describe a connection between these two realizations. In view of the semi-direct product $\bar{N} = \bar{N}_1\bar{N}_2$, $S(n,\mathbb{R}) \times \bar{n}_2 \times \mathbb{R}^n$ is identified with $\tilde{U}_g$ by the correspondence

$$S(n,\mathbb{R}) \times \bar{n}_2 \times \mathbb{R}^n \ni (X,T,t) \longmapsto \pi(g\bar{n}_1(X)\bar{n}_2(T),t) \in \tilde{U}_g,$$

where we put $\bar{n}_2(T) = \exp T$ for $T\in \bar{n}_2$. We write U^+ for U_e^+ for simplicity. For any $(T,t)\in \bar{n}_2 \times \mathbb{R}_+^n$, we put

$$Y(T,t) = (\bar{n}_2(T)a(t)^2\,{}^t\bar{n}_2(T))^{-1}.$$

Then $(X,T,t) \to X + \sqrt{-1}Y(T,t)$ defines an analytic diffeomorphism of U^+ onto $\mathcal{G}_n$.

§4. Invariant system of differential equations on X.

We introduce the Poisson kernel for the B.-S. boundary and an elliptic system of differential equations on $_n$ satisfied by the Poisson kernel.

First we define the linear form ρ' on a_2 by

$$\rho'(H_0) = n.$$

Recalling that $\rho = \sum_{i=1}^{n} (n-i+1)e_i$, as is easily seen, we have

$$\rho|_{a_2} = \frac{n+1}{2}\rho'.$$

For any s in $\mathbf{C}$, we denote by L_s the line bundle on G/P_1 associated with the character of $P_1 = M_1A_1N$; man $\to$ $\exp\{(\frac{n+1}{2} - s)\rho'(\log a)\}$. We denote by $\mathcal{B}(G/P_1;s)$ the space of all L_s-valued hyperfunctions on G/P_1. Then $\mathcal{B}(G/P_1;s)$ is canonically identified with the space of all hyperfunctions f on G such that

$$f(gman) = e^{(s-\frac{n+1}{2})\rho'(\log a)} f(g) \quad (g\in G,\ m\in M_1,\ a\in A_1,\ n\in N)$$

(cf. §2). We put

$$\lambda_s = \sum_{i=1}^{n} (s - i + \frac{n+1}{2})e_i.$$

Then it is easy to see that

$$\mathcal{B}(G/P_1;s) \subset \mathcal{B}(G/P;L(\lambda_s)).$$

For any f in $\mathcal{B}(G/P_1;s)$, we define

$$(\mathcal{J}_s f)(gK) = \int_K f(gk)dk.$$

Then the above inclusion shows that $\mathcal{J}_s f = \mathcal{J}_{\lambda_s} f$ (cf. §2) and we have

$$\mathcal{J}_s(\mathcal{B}(G/P_1;s)) \subset \mathfrak{A}(X;\mathfrak{M}(\lambda_s)).$$

The kernel function of $\mathcal{J}_s$ is, however, somewhat different from Ψ_λ. A straightforward calculation shows that

$$(\mathcal{J}_s f)(gK) = \int_K f(k)P_s(k^{-1}g)dk,$$

where

$$P_s(gK) = \exp\{-(s + \tfrac{n+1}{2})\rho'(H_0(g^{-1}))\}.^{*)}$$

If we fix any k in K, $\pi(k)P_s$ is contained in $\mathfrak{A}(X;\mathfrak{M}(\lambda_s))$. This follows from the similar argument to the proof of the same claim for $\pi(k)\Psi_\lambda$ (cf. Proposition 3.10 in [12]). Being different from Ψ_λ, $\pi(k)P_s$ can satisfy many other differential equations more than $\mathfrak{M}(\lambda_s)$. The determination of these equations is one of the interesting problems (cf. [9]). In our case, fortunately, these are easily obtained as we shall discussed below (cf. Hua [4]).

Recalling that

*) For any g G, we denote by $H_0(g)$ the unique element of a_2 such that g $K(\exp H_0(g))M_1N$.

$$K = \left\{ \begin{bmatrix} \mathrm{Re}\, U & \mathrm{Im}\, U \\ -\mathrm{Im}\, U & \mathrm{Re}\, U \end{bmatrix} ; \ U \in U(n) \right\},$$

we write $P_s(Z,U) = P_s(k^{-1}gK)$ for brevity, where

$$g = \bar{n}_1(X)a(Y),$$

$$k = \begin{bmatrix} \mathrm{Re}U & \mathrm{Im}U \\ -\mathrm{Im}U & \mathrm{Re}U \end{bmatrix} \quad (U \in U(n)).$$

Then a straightforward calculation implies that

$$P_s(Z,U) = (\det Y)^{\frac{1}{2}(s+\frac{n+1}{2})} |\det(Z\mathrm{Re}U + \mathrm{Im}U)|^{-(s+\frac{n+1}{2})}$$

(cf. Equation (4.8.6) in [4]). For any $Z = X + \sqrt{-1}Y \in \mathfrak{H}_n$, we denote by x_{ij} (resp. y_{ij}) the (i,j)-entry of X (resp. Y) and introduce the matrices:

$$\partial_X = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial x_{ij}}\right), \quad \partial_Y = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}}\right),$$

$$\partial_Z = \frac{1}{2}(\partial_X - \sqrt{-1}\partial_Y).$$

Let us introduce the differential operator

$$\Delta_Z = Y\,{}^t(Y\bar{\partial}_Z)\partial_Z$$

Lemma 1. For any $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$, we put $Z' = T_g(Z)$. Then we have

$$\Delta_{Z'} = {}^t(A + BZ)^{-1}\Delta_Z\,{}^t(A + BZ).$$

Lemma 2. $\Delta_Z P_s(Z,U) = \frac{1}{16}(s - \frac{n+1}{2})(s + \frac{n+1}{2})P_s(Z,U)$.

Lemmas 1 and 2 are proved by the same way to derive Equation (5.6.10) and Theorem 5.7.3 in [4].

Theorem 2. For any f in $\mathcal{B}(G/P_1;s)$, $u = \mathcal{J}_s f$ satisfies the differential equations;

$$16\Delta_Z u = (s - \frac{n+1}{2})(s + \frac{n+1}{2})u$$

$$16\overline{\Delta}_Z u = (s - \frac{n+1}{2})(s + \frac{n+1}{2})u.$$

The first of these follows from Lemma 2. The second is also proved by the same way.

We identify the functions on X with those on G which are right K-invariant. We denote by $\mathfrak{U}(g)$ the universal enveloping algebra of g and regard the elements of $\mathfrak{U}(g)$ as left G-invariant differential operators on G. Our next objective is to translate the above equations into those derived from $\mathfrak{U}(g)$.

Theorem 3. Let us introduce the differential operators

$$L_{ij} = L_{ji} = \sum_{k=1}^{n}(E^0_{ki}E^0_{kj} + E^-_{ki}E^-_{kj}) + (n+1)E^0_{ij} \quad (i \geqq j)$$

$$M_{ij} = M_{ji} = \sum_{k=1}^{n}(E^-_{ki}E^0_{kj} - E^0_{ki}E^-_{kj}) - (n+1)E^-_{ij}$$

Then for any f in $\mathcal{B}(G/P_1,s)$, the Poisson transform $u = \mathcal{P}_s f$ satisfies the equations

$$Du = \gamma(D)(\lambda_s)u \qquad \text{for } D \in \mathbb{D}(X),$$

$$Lu = (s - \tfrac{n+1}{2})(s + \tfrac{n+1}{2})u,$$

$$Mu = 0.$$

Here we put $L = (L_{ij})$ and $M = (M_{ij})$.

Proof. The first of these equations are obvious from the definition of $\mathcal{P}_s$. Therefore, it is sufficient to prove the rest of these. As is easily seen, the system of differential equations introduced in Theorem 3 is equivalent to the system

$$\tilde{L}u = (s - \tfrac{n+1}{2})(s + \tfrac{n+1}{2})u,$$

$$\tilde{M}u = 0,$$

where

$$\tilde{L} = 4Y({}^t(Y\partial_X)\partial_X + {}^t(Y\partial_Y)\partial_Y),$$

$$\tilde{M} = Y({}^t(Y\partial_X)\partial_Y - {}^t(Y\partial_Y)\partial_X).$$

We denote by $\tilde{L}_{ij}$ the (i,j)-entry of the matrix $\tilde{L}$.

To prove the theorem, we need a lemma which will be proved at the end of this section.

Lemma 3. For any function u on X, we have

$$(\tilde{L}_{11}u)(\sqrt{-1}I_n) = (L_{11}u)(\sqrt{-1}I_n) \quad (i = 1,\dots,n).$$

We assume this lemma for a moment. Then, in view of Lemma 1, Lemma3 shows that any solution u of the equations

$$\tilde{L}u = (s - \tfrac{n+1}{2})(s + \tfrac{n+1}{2})u,$$

$$\tilde{M}u = 0.$$

also satisfies

$$L_1 u = (s - \tfrac{n+1}{2})(s + \tfrac{n+1}{2})u$$

$$L_{i1}u = 0 \qquad (2 \leq i \leq n).$$

Since u is regarded as a right K-invariant function on G, Theorem 4 follows from these equations and the fact that

$$\{ \mathrm{Ad}(\bar{w})L_{i1} \; ; \; 1 \leq i \leq n,\ w \in W\} = \{L_{ij},\ M_{ij} \; ; \; i,j = 1,\dots,n\}.\text{*)}$$

QED

Proof of Lemma 3. For any $(a_1,\dots,a_n) \in \mathbb{R}^n_+$, we put $a = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$. In §3, we have defined two local coordinate systems on X. We assume that for any $(S,R,a) \in S(n,\mathbb{R}) \times T(n,\mathbb{R}) \times \mathbb{R}^n_+$,

*) Here we denote by W the Weyl group of (G,A) and, for any $w \in W$, $\bar{w}$ denotes an element of K which normalize A such that $\bar{w}M = w$.

$$Z(S,R,a) = X(S) + \sqrt{-1}Y(R,a) \in \mathfrak{p}_n$$

with

$$X(S) = S \quad \text{and} \quad Y(R,a) = (\exp R\cdot a^2\cdot \exp {}^tR)^{-1}.$$

A simple calculation implies that

$$Y \equiv a^{-2} - (a^{-2}R+{}^tRa^{-2}) + \frac{1}{2}(a^{-2}R^2+2{}^tRa^{-2}R+{}^tR^2a^{-2})$$

up to a matrix each entries of which is a polynomial of R whose order is higher than 2. The Jacobian of the map $(S,R,a) \to Z(S,R,a)$ are as follows:

$$\frac{\partial}{\partial x_{ij}} = \frac{\partial}{\partial s_{ij}}$$

$$\frac{\partial}{\partial y_i} = -\frac{1}{2}a_i^3\frac{\partial}{\partial a_i} + \text{(higher order terms in } R)$$

$$\frac{\partial}{\partial y_{ij}} = -a_i^2\frac{\partial}{\partial r_{ij}} + \text{(higher order terms in } R).$$

In particular, a straightforward calculation implies that

$$\frac{\partial}{\partial y_1} \equiv -\frac{1}{2}a_1^3\frac{\partial}{\partial a_1}$$

$$\frac{\partial}{\partial y_{i1}} \equiv -r_{i1}a_1^3\frac{\partial}{\partial a_1} - \sum_{1<k<i} r_{ik}a_k^2\frac{\partial}{\partial r_{k1}} - a_i^2\left(\frac{\partial}{\partial r_{i1}} + \frac{1}{2}\sum_{k>i} r_{ki}\frac{\partial}{\partial r_{k1}}\right)$$

up to polynomials of R whose orders are higher than 1.

We identify any Q in $\bar{n} + a$ with a left invariant differential operators on $\bar{N}A$ by

$$(Qu)(\bar{n}a) = \frac{d}{dt}u(\bar{n}ae^{tQ})\Big|_{t=0}.$$

Then a straightforward calculation shows that

$$E_i^0 = a_i\frac{\partial}{\partial a_i} \quad (1 \leqq i \leqq n),$$

$$E_{ij}^0 \equiv \frac{a_i}{a_j}\left(\frac{\partial}{\partial r_{ij}} + \sum_{k>i} r_{ki}\frac{\partial}{\partial r_{kj}}\right) + (\text{higher order terms in } R)$$
$$(i > j),$$

$$E_{ij}^- \equiv \frac{1}{a_i a_j}\frac{\partial}{\partial s_{ij}} + (\text{higher order terms in } R).$$

From the above equations, we obtain

$$\frac{\partial}{\partial y_1} \equiv -\frac{1}{2}a_1^2E_1^0$$

$$\frac{\partial}{\partial y_{i1}} \equiv -\frac{1}{2}a_i a_1\left(\sum_{k<i}\frac{a_k}{a_i}r_{ik}E_{k1}^0 + E_{i1}^0 - \frac{1}{2}\sum_{k>i}\frac{a_i}{a_k}r_{ki}E_{k1}^0\right).$$

(In the above discussion we have identified $a = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$ with $\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}$.)

We now proceed to prove Lemma 3. First remark that

$$\tilde{L}|_{Z=\sqrt{-1}I_n} = 4(\partial_X^2 + \partial_Y^2)|_{Z=\sqrt{-1}I_n}.$$

By the above results, we have

$$\frac{\partial}{\partial x_{ij}} = \frac{1}{2a_i a_j}E_{ij}^- + (\text{higher order terms in } R).$$

Denoting ly $\tilde{L}_{ij}^1$ the (i,j)-entry of $4\partial_X^2$, we have

$$\tilde{L}^1_{ij}\Big|_{Z=\sqrt{-1}I_n} = \sum_{k=1}^{n} (E^-_{ik}E^-_{kj})\Big|_{R=S=0,a=1}.$$

Next we calculate ∂^2_Y. By the above results, we have

$$\frac{\partial}{\partial y_{ij}} = \frac{\partial}{\partial y_{ji}} = -\frac{a_j}{2a_i}E^0_{ij} + (\text{higher order terms in } R)$$

for any i, j $(i \geq j)$ and in particular

$$\frac{\partial}{\partial y_1} \equiv -\frac{1}{2}a_1^2E_1^0,$$

$$\frac{\partial}{\partial y_{i1}} \equiv -\frac{1}{2}a_1a_i\Big(\sum_{k<i}\frac{a_k}{a_i}r_{ik}E^0_{k1} + E^0_{i1} - \frac{1}{2}\sum_{k>i}\frac{a_i}{a_k}r_{ki}E^0_{k1}\Big)$$
$$(2 \leq i \leq n)$$

up to polynomials of R whose orders are higher than 1. Hence, denoting $\tilde{L}^2_{ij}$ the (i,j)-entry of $4\partial^2_Y$, we have

$$\tilde{L}^2_{i1}\Big|_{Z=\sqrt{-1}I_n} = \Big(\sum_{k=1}^{n} E^0_{ki}E^0_{kj} + (n+1)E^0_{ij}\Big)\Big|_{R=S=0,a=1}.$$

We have thus proved the lemma.

§5. Induced system on the Martin boundary.

We first review on the theory of systems of differential equations with regular singularities. Let M be a $(n + m)$-dimensional real analytic manifold with a local coordinate system $(t_1,\ldots,t_n,x_1,\ldots,x_m)$. Set $N_j = \{t_j = 0\}$, $N = \{t_1 = \ldots = t_n = 0\}$ and $M_+ = \{t_1 > 0,\ldots, t_n > 0\}$.

We consider a system of differential equations on M

$$\mathfrak{M} : P_j u = 0 \quad (j = 1,\ldots,n).$$

Assume that $\mathfrak{M}$ has regular singularities along the set of the walls $\{N_1,\ldots,N_n\}$ with the edge N in the sense of Kashiwara-Oshima [5]. Let $s^\nu(x)\in \mathbb{C}^n$ $(\nu = 1,\ldots,r)$ denote the characteristic exponents of $\mathfrak{M}$. (We assumed $r_j = \operatorname{ord} P_j$ and $r = r_1\cdots r_n$.)

For simplicity, we assume that s^ν $(\nu = 1,\ldots,r)$ do not depend on x and that

$$\text{(A)} \qquad s^1 - s^\nu\in \mathbb{N}^n - \{0\} \quad \text{for} \quad \nu = 2,\ldots,r.$$

We need another assumption

(B) Any hyperfunction solution $u(t,x)$ of $\mathfrak{M}$ defined on M_+ has a unique extension $\tilde{u}$ on M such that

$$\operatorname{supp} \tilde{u} \subset \overline{M}_+, \qquad \tilde{u}|_{M_+} = u$$

and that

$$P_j\tilde{u} = 0 \quad (j = 1,\dots,n).$$

Then there exists a micro-differential operator $B(t,x,D_t,D_x)$ such that, for any hyperfunction solution $u(t,x)$ of $\mathcal{M}$ on M_+, we have

$$B(t,x,D_t,D_x)sp(\tilde{u}(t,x)) = sp(f(x)t_+^{s_1})^{*)}$$

with a suitable hyperfunction $f(x)$ on N.

The condition (B) is relaxed to the condition

(C) There are differential operators Q_j defined on a neighbourhood of the edge N such that $Q_j u = 0$ and, for each j $(j = 1,\dots,n)$, Q_j has regular singularities in a weak sense along the wall N_j.

For a proof of these facts and more precise statements, see [5].

We derive an important theorem concerning the induced system on N for a system with regular singularities (in a slightly extended sense). Let $\mathcal{M}$ be the system defined above. Let, further, R_k $(k = 1,\dots,p)$ be differential operators of the form

$$R_k = R_k(t,x,\vartheta,D_x),$$

where $\vartheta_j = t_j(\partial/\partial t_j)$ $(j = 1,\dots,n)$. We define a system which is a quotient of $\mathcal{M}$:

*) We call $f(x)$ the boundary-value of u in the sequel.

$$\mathfrak{N} : \begin{cases} P_j u = 0 \quad (j = 1,\dots,n), \\ \\ R_k u = 0 \quad (k = 1,\dots,p). \end{cases}$$

Put $R_k = R_k(0,x,s_1,D_x)$. Remark that $R_k = (t^{-s_1} R_k t^{s_1})\big|_{t\to 0}$. Then we have

Theorem 4.(T.Oshima) Assume (A) and (C). Then for any hyperfunction solution $u(t,x)$ of $\mathfrak{N}$ on M_+, the boundary-value $f(x)$ of u satisfies the system

$$\mathrm{Ind}(\mathfrak{N}) : R_k f = 0 \quad (k = 1,\dots,p).$$

For a proof, see T.Oshima [11].

We shall apply this theorem to the system introduced in Theorem 3. In order to describe more precisely, we define the system

$$s : \begin{cases} Du = \gamma(D)(\lambda_s)u \quad \text{for} \quad D \in \mathbb{D}(X), \\ Lu = \frac{1}{4}(s - \frac{n+1}{2})(s + \frac{n+1}{2})u, \\ Mu = 0, \end{cases}$$

for any $s \in \mathbb{C}$. The system $\mathfrak{N}_s$ is obviously a quotient of $\mathfrak{M}(\lambda_s)$. We denote by $\mathcal{A}(X,\mathfrak{N}_s)$ the space of real analytic functions on X which are solutions of $\mathfrak{N}_s$. The above remark implies that $\mathcal{A}(X,\mathfrak{N}_s) \subset \mathcal{A}(X,\mathfrak{M}(\lambda_s))$.

Then we have the following theorem.

Theorem 5. Assume $s \in \mathbb{C}$ satisfies the condition

$$\text{(D)} \qquad n - 2 - 2s \in \mathbb{N}_+ .$$

Then the Poisson transform $\mathcal{P}_s$ is a G-isomorphism of $\mathcal{B}(G/P_1, s)$ onto $\mathcal{A}(X, \mathcal{M}_s)$.

Proof. Fix $g \in G$. As we have remarked in §3, U_g^+ is identified with X and U_g^0 is an open dense submanifold of G/P. Furthermore $\tilde{U}_g \simeq \bar{N} \times \mathbb{R}^n$. We take $\lambda \in \mathfrak{a}_c^*$ satisfying the condition (A)' stated in Theorem 1. Then the system $\mathcal{M}(\lambda)$ has regular singularities along the set of walls $\{U_g^1, \ldots, U_g^n\}$ with the edge U_g^0. Here we put $U_g^i = \pi(\{(g\bar{n}, t) \in g\bar{N} \times \mathbb{R}^n;\ t_i = 0\})$ for $i = 1, \ldots, n$. (In our case, we have put $M = \tilde{U}_g$, $M_+ = U_g^+$, $N = U_g^0$, $\mathcal{M} = \mathcal{M}(\lambda)$, etc.) The characteristic exponents of $\mathcal{M}(\lambda)$ are $s_i = ((\rho - w_i\lambda)(H_1), \ldots, (\rho - w_i\lambda)(H_n))$ $(i = 1, \ldots, r)$,*) where H_j is the element of $\mathfrak{a}$ such that $\alpha_i(H_j) = \delta_{ij}$ (see, Proposition 3.4 in [6]). Under the assumption (A)', the system $\mathcal{M}(\lambda)$ satisfies the conditions (A) and (C) (see, §5 in [6]). Hence there exists a micro-differential operator $B_g(t, \bar{n}, D_t, D_{\bar{n}})$ such that, for any u in $\mathcal{A}(X; \mathcal{M}(\lambda))$, we have

$$B_g \, sp(\tilde{u}) = sp(f_g(\bar{n}) t_+^{s_1})$$

with a suitable hyperfunction $f_g(\bar{n})$ on U_g^0.

For any $s \in \mathbb{C}$, satisfying the condition (D), we take $\lambda = \lambda_s$ from now on. The condition (D) implies that λ_s satisfies (A)'. We consider the system $\mathcal{M}_s$ which is a

*) $W = \{w_1 = 1, \ldots, w_r\}$ $(r = \#W)$ denotes the Weyl group of G.

quotient of $\mathcal{M}(\lambda_s)$. Remark that

$$s_1 = (\tfrac{n+1}{2} - s, 2(\tfrac{n+1}{2} - s), \ldots, (n-1)(\tfrac{n+1}{2} - s), \tfrac{n}{2}(\tfrac{n+1}{2} - s))$$

in this case. We identify $\bar{N} \times \mathbb{R}^n$ with $\tilde{U}_g$ by the correspondence

$$\bar{N} \times \mathbb{R}^n \ni (\bar{n}, t) \longrightarrow (g\bar{n}, t) \in \tilde{U}_g.$$

Then it follows from Lemma 8 in [ ] that

$$E_i^0 \big| U_g^+ = \vartheta_{i-1} - \vartheta_i \quad (i = 1, \ldots, n-1),$$

$$E_n^0 \big| U_g^+ = \vartheta_{n-1} - 2\vartheta_n,$$

$$E_{ij}^0 \big| U_g^+ = t_j t_{j+1} \cdots t_{i-1} E_{ij}^0 \quad (i > j),$$

$$E_{ij}^- \big| U_g^+ = t_j \cdots t_{i-1} (t_i \cdots t_{n-1})^2 t_n E_{ij}^- \quad (i \geq j).$$

(Here we denote by $Y | U_g^+$ the left invariant vector field on U_g^+ corresponding to Y.) Recall the operators L_{ij} and M_{ij} (see, Theorem 3). From the above remark and the concrete expressions of L_{ij}, it follows that $(t_j \cdots t_{i-1})^{-1} L_{ij}$ is well-defined on $\tilde{U}_g$ for such i and j that $i > j$. Furthermore, a straightforward calculation impies that

$$(t^{-s_1} L_i t^{s_1}) \big|_{t \to 0} = (s - \tfrac{n+1}{2})(s + \tfrac{n+1}{2}), \quad (1 \leq i \leq n),$$

$$(t^{-s_1}(t_j\cdots t_{i-1})L_{ij}t^{s_1})\Big|_{t\to 0} = (2n+2-2s-j)E^0_{ij} + \sum_{j<k<i} E^0_{kj}E^0_{ik}$$
$$(i > j),$$
$$(t^{-s_1}M_{ij}t^{s_1})\Big|_{t\to 0} = 0.$$

Hence the induced system on U^0_g of $\mathcal{N}_s$ is

$$\mathrm{Ind}(\mathcal{N}_s) : E^0_{ij}f = 0 \quad (i > j),$$

at least assuming

(*) $$2n + 2 - j - 2s \neq 0 \quad \text{for} \quad j = 1,\ldots,n-1.$$

This system shows that the boundary value $f_g(\bar{n})$ of any u $\mathcal{A}(X,\mathcal{N}_s)$ is invariant under the right action of $\bar{N}_2$. In case when s satisfies the condition (*), we replace u by $t^k u$ for a large positive number k. Then from Lemma 5.13 in [ ] one can also derive the induced system $\mathrm{Ind}(\mathcal{N}_s)$. Hence (*) can be omitted.

As proved in Proposition 2.28 in [12], for any $u \in \mathcal{A}(X,\mathcal{M}(\lambda_s))$, the boundary-value $f_g(\bar{n})$ defines an element of $\mathcal{B}(G/P,L(\lambda_s))$. More precisely, $f_g(\bar{n})(dt)^{\otimes s_1}$ does not depend on $g\in G$ and is contained in $\mathcal{B}(G/P;L(\lambda_s))$. Hence the above argument shows that, if u is contained in $\mathcal{A}(X;\mathcal{N}_s)$, the "boundary-value" $f_g(\bar{n})(dt)^{\otimes s_1}$ is an element of $\mathcal{B}(G/P;L(\lambda_s))$ which is invariant under the right action of $\bar{N}_2$, namely, is contained in $\mathcal{B}(G/P_1;s)$. We have thus defined a G-homomorphism of $\mathcal{A}(X,\mathcal{N}_s)$ into $\mathcal{B}(G/P_1;s)$ (cf. §4 in [6]).

We now proceed to the proof of the theorem. Under the assumption (D), it follows from Theorem 1 that $\mathfrak{I}_{\lambda_s}$ defines a G-isomorphism of $\mathcal{B}(G/P, \mathcal{L}(\lambda_s))$ onto $\mathcal{Q}(X, \mathfrak{M}(\lambda_s))$. We have expressed an inverse of $\mathfrak{I}_{\lambda_s}$ by use of the theory of systems of differential equations with regular singularities. As we have remarked in §4 that $\mathfrak{I}_{\lambda_s}$ maps any element of $\mathcal{B}(G/P_1, s)$ into $\mathcal{Q}(X; \mathfrak{N}_s)$. On the other hand, the boundary-value of any element of $\mathcal{Q}(X, \mathfrak{N}_s)$ is contained in $\mathcal{B}(G/P_1, s)$ as we have proved above. Hence the theorem. Q D

Remark: In a subsequent paper, it will be proved that the system $\mathfrak{N}_s$ is equivalent to the system

$$\mathfrak{N}'_s : \begin{cases} Lu = \left(s - \frac{n+1}{2}\right)\left(s + \frac{n+1}{2}\right)u, \\ \\ Mu = 0. \end{cases}$$

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