ユニタリ表現論セミナー報告集 VII, 1987
pp.81-96

Two Remarks on Recent Development in Solvable Models

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1. Introduction

In the exact study of two dimensional statistical mechanics and quantum field theory, two approaches are presently available: solvable lattice models (SLMs) [1] and conformal field theory (CFT) [2]. The ideas and methods in these approaches are quite independent of each other. On the one hand, the SLM deals with generically non-critical models on the lattice, and is built upon solutions to a set of algebraic equations called the star-triangle relation. The CFT on the other hand deals with critical and continuous systems. The main tool here is the symmetry under infinite dimensional Lie algebras, notably the Virasoro algebra.

Notwithstanding their conceptual difference these theories exhibit unexpected similarities. On the basis of the recent works on SLM [3-9], we point out in this article two similar structures that occur in different contexts: One is the representation of the braid group and the other is the modular covariance. The true nature of these phenomena is still unknown. We believe that once understood properly it will shed new light to the theory of integrable systems.

2. Star-triangle relation and the braid group

In [4,8,9] we have constructed a family of solvable lattice models labeled by three positive integers n, ℓ, N with $\ell > N$. Leaving aside its physical content, let us briefly explain the setting in order to make contact with the braid group.

Consider partitions $(f_0, \cdots, f_{n-1})$, $f_i \in \mathbb{Z}$, satisfying

$$f_0 \geq \cdots \geq f_{n-1} \geq 0, \quad f_0 - f_{n-1} \leq \ell.$$

Denote by $P_+(n;\ell)$ the set of their equivalence classes under the relation $(f_0, \cdots, f_{n-1}) \sim (f_0+1, \cdots, f_{n-1}+1)$. Let Λ_μ $(0 \leq \mu \leq n-1)$ denote the fundamental weights of the affine Lie algebra $A^{(1)}_{n-1}$. An element $a = [f_0, \cdots, f_{n-1}]$ of $P_+(n;\ell)$ is identified with a level ℓ dominant integral weight $\sum_{\mu=0}^{n-1}(f_{\mu-1}-f_\mu)\Lambda_\mu$ $(f_{-1} = \ell - f_0 + f_{n-1})$. If $a, b \in P_+(n;\ell)$ are related by $a = [f_0, \cdots, f_{n-1}]$ and $b = [f_0, \cdots, f_\mu+1, \cdots, f_{n-1}]$ with some $\mu = 0, 1, \cdots, n-1$, we write $b = a + \hat{\mu}$. An ordered pair (a, b) is said to be N-admissible if there exist $\xi \in P_+(n;\ell-N)$ and $\eta \in P_+(n;N)$ such that

$$a = \xi + \eta, \quad b = \xi + \sigma(\eta), \tag{1}$$

where σ denotes the cyclic diagram automorphism $\sigma(\Lambda_\mu) = \Lambda_{\mu+1}$. For $N = 1$ (1) simply means $b = a + \hat{\mu}$ with some μ. Now let $\mathcal{P}_m$ be the set of 'paths' consisting of N-admissible pairs

$$\mathbf{a} = (a_1, \cdots, a_m), \quad (a_i, a_{i+1}) \text{ is N-admissible for } 1 \le i \le m-1.$$

Let $V_m = \oplus \mathbf{C}v_{\mathbf{a}}$ be the span of orthonormal vectors $v_{\mathbf{a}}$ indexed by $\mathbf{a} \in \mathcal{P}_m$. By a local face operator we mean a matrix U_i $(2 \le i \le m-1)$ that acts on V_m in the following manner:

$$U_i v_{\mathbf{a}} = \sum_{\mathbf{a}'} W\begin{pmatrix} a_{i-1} & a_i \\ a'_i & a_{i+1} \end{pmatrix} v_{\mathbf{a}'}.$$

Here $\mathbf{a}'$ runs over the paths that differ by $\mathbf{a}$ only at the i-th component a_i. The scalar coefficients $W\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ represent the Boltzmann weights in the language of statistical mechanics. Clearly $U_i U_j = U_j U_i$ if $|i-j| \ge 2$.

The algebraic content of the construction of SLM amounts to the following result. Suppose U_i depends on an extra variable $u \in \mathbf{C}$.

Theorem.[4,8,9] For each n, ℓ, N there exist face operators $U_i(u)$ acting on V_m such that the star-triangle relation (STR) is valid:

$$U_i(u)U_{i+1}(u+v)U_i(v) = U_{i+1}(v)U_i(u+v)U_{i+1}(u), \quad u, v \in \mathbb{C}. \tag{2}$$

The STR guarantees the commutativity of the transfer matrix for different values of u [1]. Our solution is parametrized by an elliptic theta function

$$\theta_1(u,p) = 2|p|^{1/8}\sin u \prod_{k=1}^{\infty}(1-2p^k\cos 2u + p^{2k})(1-p^k).$$

For instance in the case $N = 1$, the $U_i(u)$ are given explicitly by

$$\begin{aligned} W\begin{pmatrix} a & a+\hat\mu \\ a+\hat\mu & a+2\hat\mu \end{pmatrix}\!\Big|\, u\Big) &= \frac{[1+u]}{[1]}, \\ W\begin{pmatrix} a & a+\hat\mu \\ a+\hat\mu & a+\hat\mu+\hat\nu \end{pmatrix}\!\Big|\, u\Big) &= \frac{[a_{\mu\nu}-u]}{[a_{\mu\nu}]}, \\ W\begin{pmatrix} a & a+\hat\nu \\ a+\hat\mu & a+\hat\mu+\hat\nu \end{pmatrix}\!\Big|\, u\Big) &= \frac{[u]}{[1]}\frac{[a_{\mu\nu}+1]}{[a_{\mu\nu}]}, \end{aligned} \tag{3}$$

where $[u] = \theta_1(\pi u/L, p)$, $L = \ell + n$, and $a_{\mu\nu} = f_\mu - f_\nu + \nu - \mu$ for $a = [f_0, \cdots, f_{n-1}]$. The models with $n = 2$, $N = 1$ have been introduced and studied in detail by Andrews-Baxter-Forrester [10] under the name 'eight vertex SOS models'.

The models become critical at $p = 0$. The local face operators U_i then reduce to polynomials in the variable $x = e^{2\pi i u/L}$

$$const.\ U_i(u) = g_i x^N + g'_i x^{N-1} + \cdots + g''_i \ , \tag{4}$$

and the STR (2) together with the the commutativity $U_i U_j = U_j U_i$ ($|i-j| \geq 2$) imply that their leading coefficients g_i give a representation of the braid group on V_m:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if} \quad |i-j| \geq 2.$$

In the case $N = 1$ we have an additional relation

$$(g_i - 1)(g_i + q) = 0, \qquad q = e^{2\pi i/L},$$

which means that the representation factors through the Iwahori -Hecke algebra. In fact, when restricted to the subspace V_m^ϕ = the linear span of $\{v_a \mid a_1 = [0, 0, \cdots, 0]\}$, formulas (3), (4) with $[u]$ replaced by $\sin(\pi u/L)$ reproduce the irreducible representation studied by Wenzl [11]. (For $n = 2$ this has been noted in [12].)

A very similar structure has been encountered by Tsuchiya-Kanie [13] in their study of the CFT with the gauge symmetry with respect to $A_1^{(1)}$. Let V_j (resp. $\mathcal{H}_j$) denote the irreducible A_1 module (resp. $A_1^{(1)}$ module) of spin $j = 0, \frac{1}{2}, 1, \cdots, \frac{\ell}{2}$, *i.e.* the one with highest weight $2j\Lambda_1$ (resp. $(\ell-2j)\Lambda_0 + 2j\Lambda_1$). Put $\mathcal{H} = \bigoplus_{0 \leq j \leq \ell/2} \mathcal{H}_j$. It is known that on $\mathcal{H}$ there is a natural action of the Virasoro algebra $\mathcal{Vir} = (\bigoplus_{n \in Z} CL_n) \oplus Cc$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\ \delta_{m+n,0}, \qquad [L_m, c] = 0. \tag{5}$$

Given a triple $v = \begin{pmatrix} & j & \\ j_2 & & j_1 \end{pmatrix}$ we consider an operator $\Phi_v(u;z)$ on $\mathcal{H}$, linearly parametrized by $u \in V_j$ and satisfies $\pi_{j_2}\Phi_v(u;z)\pi_{j_1} = \Phi_v(u;z)$ where π_j signifies the projection $\mathcal{H} \longrightarrow \mathcal{H}_j$. It is called a vertex operator if

$$[X\otimes t^m, \Phi_v(u;z)] = z^m\Phi_v(Xu;z), \qquad X\otimes t^m \in A_1^{(1)},$$
$$[L_m, \Phi_v(u;z)] = z^m(z\frac{d}{dz} + (m+1)\Delta_j)\Phi_v(u;z), \quad \Delta_j = \frac{j(j+1)}{\ell+2}.$$

In the above we identified $A_1^{(1)}$ with $sl(2,\mathbb{C})\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\ell$. A nontrivial vertex operator exists if and only if [13]

$$|j_1-j_2| \le j \le j_1+j_2, \ j \equiv j_1+j_2 \bmod \mathbb{Z} \ \text{ and } \ j_1+j_2+j \le \ell. \tag{6}$$

In our terminology this condition states that $b = [2j_1,0]$ and $c = [2j_2,0] \in P_+(2;\ell)$ are 2j-admissible. Tsuchiya-Kanie derived differential equations and supplementary algebraic relations for the correlation functions $\langle\Phi_{v_m}(u_m;z_m)\cdots\Phi_{v_1}(u_1;z_1)\rangle$, and showed that those equations admit a basis of solutions (in fact the correlations themselves) indexed by the set $\{ p = (p_m,\cdots,p_0) \mid p_i\in \frac{1}{2}\mathbb{Z},\ 0\le p_i\le \frac{\ell}{2},\ p_m = p_0 = 0$ and $v_i = \begin{pmatrix} & j_i & \\ p_i & & p_{i-1} \end{pmatrix}$ satisfies (6)$\}$. When $j_1=\cdots=j_m= \frac{N}{2}$, the corresponding monodromy representation realizes the braid relation. They verified that in the case $N = 1$ there results Wenzl's representation of the

Iwahori-Hecke algebra for $q = e^{2\pi i/(\ell+2)}$. Similar result has been obtained also by Kohno [14] when q is not a root of unity.

It is noteworthy that the same representation appears in totally different contexts.

3. Modular covariant characters

Our second remark is on different roles of modular covariance in solvable models.

Kac-Peterson [15] established the close connection between the characters of affine Lie algebras, theta functions and modular forms. Of significance here is an object called branching coefficients. Let $(\mathfrak{g}, \mathfrak{h})$ be a pair of affine Lie algebra and its subalgebra. Consider the irreducible decomposition of a highest weight $\mathfrak{g}$-module $L^{\mathfrak{g}}(\Lambda)$ with respect to $\mathfrak{h}$:

$$L^{\mathfrak{g}}(\Lambda) = \sum \Omega_{\Lambda\Lambda'} \otimes L^{\mathfrak{h}}(\Lambda').$$

Rewriting it as an identity of characters one gets

$$\chi^{\mathfrak{g}}_{\Lambda} = \sum_{\Lambda'} b_{\Lambda\Lambda'}(q)\, \chi^{\mathfrak{h}}_{\Lambda'},$$

which defines the banching coefficients $b_{\Lambda\Lambda'}(q)$. Goddard-Kent-Olive [16] provided a method to construct a representation of the Virasoro algebra (5) on the space $\Omega_{\Lambda\Lambda'}$.

The $b_{\Lambda\Lambda'}(q)$ can be regarded as its characters. Thanks to the transformation formula for the characters $\chi^{\mathfrak{g}}_{\Lambda}$, $\chi^{\mathfrak{h}}_{\Lambda'}$, the branching coefficients also enjoy automorphic properties.

In SLM the branching coefficients appear as the local state probabilities (LSPs). By an LSP we mean the probability that the local fluctuation variable a_1 on a lattice site 1 takes a given state a: $P_a = \mathrm{prob}(a_1 = a)$.

Consider the models in sect. 2 in the region $0 < p < 1$, $-n/2 < u < 0$, and put $p = e^{2\pi i\tau}$, $q = e^{-2\pi in/L\tau}$. For $a \in P_+(n;\ell)$ we denote by $L(a)$ the irreducible $A^{(1)}_{n-1}$ module with highest weight a. Let $(\mathfrak{g}, \mathfrak{h}) = (A^{(1)}_{n-1} \oplus A^{(1)}_{n-1}, A^{(1)}_{n-1})$, and denote by $b_{\xi\eta a}(q)$ the branching coefficients associated with the decomposition of the tensor product $L(\xi)\otimes L(\eta)$, $\xi \in P_+(n;\ell-N)$, $\eta \in P_+(n;N)$, with respect to the diagonal subalgebra $A^{(1)}_{n-1}$.

Theorem. [5-8] Suppose $n = 2$ or $N = 1$. Then the LSP is given by

$$P_a = b_{\xi\eta a}(q)\ \chi^{sp}_a \,/\, \chi^{sp}_{\xi}\ \chi^{sp}_{\eta}, \tag{7}$$

where χ^{sp}_a signifies the principally specialized character of $L(a)$, and ξ, η are related to the choice of the boundary condition.

For a more precise statement, see [7, 8].

In the evaluation of the LSP the crucial tool is Baxter's corner transfer matrix (CTM) defined as follows. Divide the lattice into four quadrants A, B, C, D.

[Fig. 1]

Choose the states on the two axes $a = (\lambda_1, \cdots, \lambda_m)$ and $a' = (\mu_1, \cdots, \mu_m)$. We denote by $\mathcal{A}_{aa'}$ the partition function for the quadrant A with this boundary condition, the sum being taken over the internal variables (open circles in Fig. 1). The CTM $\mathcal{A}$ is the matrix with entries $\mathcal{A}_{aa'}$. Other CTMs $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are defined similarly. They are block diagonal in the sense $\mathcal{A}_{aa'} = 0$ if $\lambda_1 \neq \mu_1$. Let e^{-E_j} $(j = 0, 1, \cdots)$ be the eigenvalues of $\mathcal{A}$ in a fixed block $\lambda_1 = \mu_1 = a$. Baxter found in the solvable cases the following behavior of E_j as $m \longrightarrow \infty$:

$$E_j - E_0 \sim \mathit{const.}\, N_j u, \qquad N_j : \text{integer}, \tag{8}$$

where u is the 'spectral parameter' entering in the STR (2). The heart of the result (7) is that (up to an overall shift) the N_j constitute the spectrum of L_0 in the space $\Omega_{\Lambda\Lambda'}$. The automorphic properties of the $b_{\xi\eta a}(q)$ and of the specialized characters afford us the complete knowledge about the behavior of the LSP in the critical limit $p \longrightarrow 0$, $q \longrightarrow 1$.

Another modular invariance shows up when one considers lattice systems at criticality. Consider a lattice $\mathcal{L}_n$ wound on a cylinder by the periodic boundary condition $(i_1+n, i_2) = (i_1, i_2)$ (Fig. 2).

[Fig. 2]

Let e^{-H} be the transfer matrix in the vertical direction of a *critical* statistical model on $\mathscr{C}_n$. We denote by V_n the space on which acts H, and denote by e^{ik} the horizontal shift operator on V_n. Let E_j $(j=0,1,\cdots)$ be eigenvalues of H. The criticality of the model implies the following scaling behaviors of the lowest eigenvalue E_0 and of low lying excitations (*cf.* (8)) as the width of the cylinder n becomes large:

$$E_0 \sim fn - \frac{\pi c}{6n}$$
$$E_j - E_0 \sim \frac{2\pi x_j}{n} \qquad x_j:\ \text{integer}. \tag{9}$$

On the basis of the conformal invariance of critical systems, Cardy [17] related the constants c, x_j to the representation of the Virasoro algebra. Let V be the space, considered in the limit of $n \longrightarrow \infty$, spanned by the vectors corresponding to eigenvalues satisfying (9). Cardy's assertion is the following: There exist *two* Virasoro operators $\{L_n\}$ and $\{\bar{L}_n\}$, mutually commutative and acting on V with the common value of the central charge c. The operators L_0 and $\bar{L}_0$ are related to H and k by

$$H - E_0 \sim \frac{2\pi(L_0+\bar{L}_0)}{n}, \qquad k \sim \frac{2\pi(L_0-\bar{L}_0)}{n}. \tag{10}$$

Next consider the lattice $\mathcal{T}_{mn}$ on the torus obtained by imposing further the twisted boundary condition $(i_1, i_2+m') = (i_1+m'', i_2)$ in the vertical direction $(m = m'+\sqrt{-1}\,m'')$. We introduce $\tau = \sqrt{-1}\,m/n$ as the moduli parameter of the torus. The finite part $Z(\tau)$ of the partition function is defined by the following in the limit $m, n \longrightarrow \infty$ with τ fixed:

$$Z(\tau) = \lim \frac{Z_{mn}}{e^{-fm'n}} = \lim \operatorname{trace}_{V_n}(e^{-m'(H-fn)} e^{-im''k}),$$

where Z_{mn} is the partition function on $\mathcal{T}_{mn}$. By (10) it is written as a sesqui-linear form of Virasoro characters $\psi_{ch}(q)$:

$$\begin{aligned} Z(\tau) &= \operatorname{trace}_V(q^{L_0-c/24} q^{*\bar{L}_0-c/24}) \\ &= \sum_{(h,\bar{h})} N_{h,\bar{h}}\, \psi_{c,h}(q)\psi_{c,\bar{h}}(q^*), \end{aligned}$$

where $q = e^{2\pi i\tau}$, q^* is the complex conjugate, and $N_{h,\bar{h}}$ are nonnegative integers.

By the definition it is clear that $Z(\tau+1) = Z(\tau)$. In the non twisted case $m'' = 0$ (*i.e.* when τ is purely imaginary), one must have $Z(-1/\tau) = Z(\tau)$ because of the invariance under the rotation of the lattice through 90 degrees. For the Ising model this relation has been verified directly by Ferdinand-Fisher [18] via an exact computation of $Z(\tau)$. If one accepts the invariance of $Z(\tau)$ under the full modular group

$SL_2(\mathbb{Z})$, the possible choice of V is strongly restricted.

When V decomposes into a direct sum of a finite number of irreducible representations of $\mathit{Vir}\oplus\overline{\mathit{Vir}}$, the theory is called minimal. As an example, consider the eight vertex SOS models (the case $n = 2$ and $N = 1$). In this case the branching coefficients $b_{\xi\eta a}(q)$ give irreducible unitary characters [19] of Vir with the central charge

$$c = 1 - \frac{6}{L(L-1)} < 1.$$

Set $\xi = (L-2-s)\Lambda_0 + (s-1)\Lambda_1$, $a = (L-1-t)\Lambda_0 + (t-1)\Lambda_1$, $\eta = \Lambda_0$ or Λ_1 according as $s \equiv t$ or $t+1 \bmod 2$. Put further

$$h = h_{st} = \frac{(Ls-(L-1)t)^2-1}{4L(L-1)}, \qquad 1 \le t \le s \le L-2.$$

We have then

$$\begin{aligned} b_{\xi\eta a}(q) &= \psi_{c,h_{st}}(q) \\ &= \eta(\tau)^{-1} \sum_{\varepsilon=\pm 1} \varepsilon \sum_{\gamma\in\mathbb{Z}+s/2(L-1)-\varepsilon t/2L} q^{L(L-1)\gamma^2} \end{aligned}$$

where $\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j)$, $q = e^{2\pi i\tau}$. It is argued [17] that the torus partition function is given by

$$Z(\tau) = \sum_{1\le t\le s\le L-2} |\psi_{c,h_{st}}(q)|^2.$$

The branching coefficients used in the LSP result arise here as the constituents of $Z(\tau)$. The complete list of the torus partition functions of the minimal conformal theories is available [20,21]. In the case of non-minimal theories the finite reducibility with respect to larger symmetry algebras play similar crucial roles [22-26].

There are other examples where the coincidence is established of the LSP and the branching coefficients for appropriate choice of the pair $(\mathfrak{g}, \mathfrak{h})$ ([3],[6]). In these cases the corresponding torus partition functions are also supposed to be given in terms of the same set of branching coefficients. This coincidence is quite mysterious because the origins of the torus are totally different. In Cardy's theory the torus appears as the continuum limit of the finite lattice, while it appears as the one that describes the commuting family of transfer matrices of SLM. We expect that there is a unified way of understanding conformal field theory, solvable lattice models and the role of infinite dimensional Lie algebras.

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