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Hecke代数と D加群

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1. Some problems in the representation theory of reductive groups are closely connected with various geometric objects related to these groups, and these geometric pictures are often well understood through the Weyl groups, the affine Weyl groups and their Hecke algebras.

Examples.

(i) The character formula of irreducible admissible modules of reductive groups (Kazhdan-Lusztig [9], Brylinski-Kashiwara [3], Beilinson-Bernstein [1]).

(ii) The Springer representations (Springer [16], Hotta-Springer [5], Shoji [15],...).

(iii) The classification of the irreducible modules of the Hecke algebra of an affine Weyl group (Kazhdan-Lusztig [11], Ginsburg [4]).

Example (ii) is related to the computation of the values of irreducible characters of finite reductive groups at unipotent elements (the Green polynomials), and (iii) is equivalent to the classification of the irreducible admissible modules of reductive p-adic groups containing non-zero vectors fixed by an Iwahori subgroup (Iwahori-Matsumoto [7], Matsumoto [13], Borel [2]).

One of the purposes of this note is to clarify the relations of the above examples.

2. Let us explain (i) and (ii) from our point of view.

(i) Let G be a complex reductive group and X its flag variety. We denote by $\tilde{A}$ the category of coherent $D_{X\times X}$-modules with (diagonal) G-actions. Here $D_{X\times X}$ is the sheaf of (algebraic) differential operators on $X\times X$. We have certain standard $D_{X\times X}$-modules $\tilde{M}_w$ and $\tilde{L}_w$ in $\tilde{A}$, where w is an element of the Weyl group W. $\{ [\tilde{M}_w] \}_{w\in W}$ and $\{ [\tilde{L}_w] \}_{w\in W}$ are free $\mathbf{Z}$-bases of the Grothendieck group $K(\tilde{A})$. We identify $K(\tilde{A})$ with the group ring $\mathbf{Z}[W]$ via $[\tilde{M}_w] \longleftrightarrow w$ and denote the element of $\mathbf{Z}[W]$ corresponding to $[\tilde{L}_w]$ by $a(w)$.

By the Beilinson-Bernstein categorical equivalence [1], $\tilde{A}$ is equivalent to a category of certain representations of G. Then $\tilde{M}_w$ corresponds to a principal series module and $\tilde{L}_w$ corresponds to the unique simple quotient of the principal series module. Since characters of principal series modules are known, the problem is reduced to computing the $a(w)$'s.

Let $H(W)$ be the Hecke algebra of W (see Iwahori [6] and Kazhdan-Lusztig [9]). It is an algebra over the Laurent polynomial ring $\mathbf{Z}[q,q^{-1}]$ with a standard basis $\{ T_w \}_{w\in W}$. Via the specialization $q \to 1$, we have :

$$\begin{array}{ccc} H(W)|_{q=1} & \simeq & \mathbb{Z}[W] \\ \Psi & & \Psi \\ (-1)^{\ell(w)} T_w|_{q=1} & \longleftrightarrow & w\ , \end{array}$$

where ℓ is the length function. Kazhdan-Lusztig [9] defined a different basis $\{ C_w^* \}_{w \in W}$ of $H(W)$ and conjetured that $a(w) = (-1)^{\ell(w)} C_w^*|_{q=1}$. This conjecture was settled by Beilinson-Bernstein [1] and Brylinski-Kashiwara [3].

(ii) Let Z be the Steinberg triple variety, that is,

$$Z = \{ (A,x,y) \in \mathrm{Lie}(G)\times X\times X \mid A \text{ is nilpotent, } A \in \underline{b}_x \cap \underline{b}_y \}.$$

Here $\underline{b}_x$ is the Borel subalgebra of $\mathrm{Lie}(G)$ corresponding to $x \in X$. Z has pure dimension $d = 2\dim X$ and the irreducible components of Z are parametrized by W. Let Z_w be the irreducible component of Z corresponding to $w \in W$.

Springer and Kazhdan-Lusztig defined a $W\times W$-module structure on the top homology group $H_{2d}(Z) = \bigoplus_{w\in W} \mathbb{Z}[Z_w]$ and showed that it is isomorphic to $\mathbb{Z}[W]$ as a $W\times W$-module via the correspondence $[Z_e] \leftrightarrow e$, where e is the identity element of W (see Kazhdan-Lusztig [10]). Let $b(w)$ be the element of $\mathbb{Z}[W]$ corresponding to $[Z_w]$. Since the basis $\{ b(w) \}_{w\in W}$ of $\mathbb{Z}[W]$ contains much information concerning the Springer representation, it is important to find a method for computing the $b(w)$'s explicitly.

3. In Kashiwara-Tanisaki [8] we constructed an intertwining operator :

$$(\#) \qquad \underline{\underline{\mathrm{Ch}}} : K(\tilde{A}) \to H_{2d}(Z)$$

of $W\times W$-modules using the characteristic cycles of D-modules and showed that it is compatible with the identifications of $K(\tilde{A})$ and $H_{2d}(Z)$ with $\mathbb{Z}[W]$ explained in Section 2. In consequence, if we write $a(w) = \sum_{y\in W} m(y,w)b(y)$, the integer $m(y,w)$ coincides with the multiplicity of the characteristic cycle of the D-module $\tilde{\mathcal{L}}_w$ along Z_y. We deduced some properties of the Springer representation from this analytic description of $m(y,w)$.

Although the algorithm for computing the $m(y,w)$'s (hence the $b(y)$'s) is not yet known, we have got some partial results in [18].

Besides, we have generalized the above result of [8] in [17] and [19]. In [17] we have considered real reductive groups and in [19] we have used an affine Weyl group instead of the Weyl group.

4. In this note we generalize the result of [8] using $H(W)$ instead of $\mathbb{Z}[W]$ (see [20]).

We first give a realization of $H(W)$ using M. Saito's theory of Hodge modules (see [14]). Let A_0 be the category consisting of the Hodge modules on $X\times X$ with G-actions. We have certain standard objects M_w and L_w of A_0 for $w \in W$ and the underlying $D_{X\times X}$-modules of M_w and L_w coincide with $\tilde{M}_w$ and $\tilde{L}_w$. Let A be the full subcategory of A_0 whose objects have composition factors of the form $L_w(n)$ for some $w \in W$ and $n \in \mathbf{Z}$, where (n) is the Tate twist. We can define a $Z[q,q^{-1}]$-algebra structure on the Grothendieck group $K(A)$ and we have the following :

Theorem A ([20]). $K(A)$ is isomorphic to $H(W)$ as a $\mathbf{Z}[q,q^{-1}]$-algebra. The isomorphism is given by $[M_w] \longleftrightarrow (-1)^{\ell(w)}T_w$ and $[L_w] \longleftrightarrow (-1)^{\ell(w)}C_w^*$.

Recently Kazhdan-Lusztig [11] and Ginsburg [4] have identified the equivariant K-homology group $K^{G\times\mathbf{C}^*}(Z)$ of the Steinberg triple variety with the Hecke algebra $H(W_a)$ of the affine Weyl group W_a, and have obtained irreducible modules of the Hecke algebra decomposing this space (Example (iii)).

Analogous to (#) we have a natural map :

$$(\#\#) \quad gr : K(A) \to K^{G\times\mathbf{C}^*}(Z)$$

defined in terms of Hodge modules.

Theorem B ([20]). If we identify $K(A)$ and $K^{G\times\mathbf{C}^*}(Z)$ with $H(W)$ and $H(W_a)$ respectively, then the map gr coincides with the natural inclusion.

This theorem gives a natural explanation of the fact that the Hecke algebra appears in the context of the equivariant K-theory. Furthermore it gives an alternative proof of the above mentioned result of Kazhdan-Lusztig and Ginsburg.

Another motivation of our result is a conjecture of Lusztig [12] which claims that the two-sided cells of W_a are in one-to-one correspondence with the nilpotent orbits in Lie(G). In order to prove the conjecture it should be important to specify the element of $K^{G\times\mathbb{C}^*}(Z)$ corresponding to C_w^* for each $w \in W_a$. Our result gives a description of this element for $w \in W$.

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