

Discrete Part of L^p Functions on $SU(1,1)$

By

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§1. Introduction. Let G be $SU(1,1)$ and T_n ($n \in \frac{1}{2}\mathbb{Z}$, $|n| \geq 1$) the holomorphic ($n > 0$) and the anti-holomorphic ($n < 0$) discrete series representation of G (cf. [Su], p.237). Let $f_{\ell m}^n$ ($\ell, m \in \mathbb{N}$) denote the normalized matrix coefficient of T_n with K -type $(-(n \pm \ell), -(n \pm m))$, where " $\pm$ " corresponds to the $\pm$ sign of n . Then the Fourier coefficients of f in $L^2(G)$ are defined by

$$\hat{f}(n; \ell, m) = \int_G f(g) \overline{f_{\ell m}^n(g)} dg \quad (1.1)$$

for $n \in \frac{1}{2}\mathbb{Z}$, $|n| \geq 1$ and $\ell, m \in \mathbb{N}$. In the previous manuscript [K] we define the operator ϕ_m^n which maps the function f on G to the function $\phi_m^n(f)$ on D , the open unit disk in $\mathbb{C}$, as follows.

$$\begin{aligned} \phi_m^n(f)(z) &= \int_G \overline{f(g)} T_n(g) e_m^n(z) dg \quad (z \in D) \\ &= 2\pi^{\frac{1}{2}} (2n-1)^{-\frac{1}{2}} \sum_{\ell=0}^{\infty} \overline{\hat{f}(n; \ell, m)} e_n^{\ell}(z), \end{aligned} \quad (1.2)$$

where $e_n^{\ell}(z) = (\Gamma(\ell+2n)/\Gamma(\ell+1)\Gamma(2n-1))^{\frac{1}{2}} z^{\ell}$ ($\ell \in \mathbb{N}$). Then a characterization of $\phi_0^n(L^p(G))$ ($1 \leq p \leq 2$) is obtained in [K], Theorem 8.1, actually, if $(n, p) \neq (1, 1)$, it coincides with the weighted Bergman space $A_{p, \frac{1}{2}np-1}(D)$ on D (cf. [CR]), and if $(n, p) = (1, 1)$, it is given by the subspace $H_0^1(D)$ of the classical Hardy space $H^1(D)$ which consists of all holomorphic functions F on D satisfying $F' \in A_{1,0}(D)$.

However, the approach and their proofs in [K] are complicated, and we treat only the case of $m=0$. Therefore, in this manuscript, we shall try to reform them and obtain the independence of m for $\phi_m^n(L^p(G))$. Moreover, we shall give some applications to the classical harmonic analysis on D .

§2. Notation. We shall use the same notations in [K]. For the readers who read first we shall brief them.

2.1. L^p functions on $SU(1,1)$. Let G be $SU(1,1)$, the group of 2×2 complex matrices g satisfying $\det g = 1$ and $g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and let

$$\begin{aligned} K = \{ k_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} ; 0 \leq \theta \leq 4\pi \}, \\ A = \{ a_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix} ; t \in \mathbb{R} \}. \end{aligned} \quad (2.1)$$

For a complex valued function f on G we put

$$\|f\|_p = \left(\int_G |f(g)|^p dg \right)^{1/p} \quad (0 < p < \infty), \quad (2.2)$$

where dg is the Haar measure on G normalized by the following integral formula for the Cartan decomposition $G = KCL(A^+)K$ ($A^+ = \{ a_t ; t > 0 \}$):

$$\int_G f(g) dg = \frac{1}{8\pi} \int_0^{4\pi} \int_0^\infty \int_0^{4\pi} f(k_\theta a_t k_{\theta'}) sht d\theta dt d\theta', \quad (2.3)$$

where $d\theta$, $d\theta'$ and dt denote the Euclidean measures on $[0, 4\pi)$ and $\mathbb{R}$ respectively. Then $L^p(G)$ ($0 < p < \infty$) denotes the space of all complex valued functions on G with finite $\|\cdot\|_p$ -norm. For $1 \leq p < \infty$, these spaces are Banach spaces with the norm, especially, $L^2(G)$ is a Hilbert space with obvious inner product denoted by $(\cdot, \cdot)$.

2.2. Bergman and Hardy spaces on D . Let $D = \{ z \in \mathbb{C} ; |z| < 1 \}$ be the open unit disk in $\mathbb{C}$. For a complex valued function F on D we put

$$\begin{aligned} \|F\|_{p,r} &= \left(\frac{1}{\pi} \int_D |F(z)|^p (1-|z|^2)^{2r} dz \right)^{1/p} \quad (0 < p < \infty, r \in \mathbb{R}) \\ \|F\|_{H^p} &= \lim_{r \rightarrow -\frac{1}{2}} (2r+1) \|F\|_{p,r} \quad (0 < p < \infty), \end{aligned} \quad (2.4)$$

where dz is the Euclidean measure on D . Then the weighted Bergman space $A_{p,r}(D)$ and the Hardy space $H^p(D)$ are defined as the spaces of all holomorphic

functions on D with finite $\| \cdot \|_{p,r}$ and $\| \cdot \|_{H^p}$ - "norms" respectively. For $1 \leq p < \infty$, these spaces are Banach spaces with the norms, especially, if $p=2$, they are Hilbert spaces with obvious inner products. Moreover, it is easy to see that when $r \leq -\frac{1}{2}$, $A_{p,r}(D) = \{0\}$, and each $\|F\|_{H^p}$ is also given by

$$\|F\|_{H^p} = \sup_{0 < r < 1} \left(\frac{1}{\pi} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta \right)^{1/p}. \quad (2.5)$$

For $r > -\frac{1}{2}$ we put

$$e_{r+1}^\ell(z) = B(\ell+1, 2r+1) z^\ell \quad (\ell \in \mathbb{N}). \quad (2.6)$$

Then $\{e_{r+1}^\ell; \ell \in \mathbb{N}\}$ is a complete orthonormal basis of $A_{p,r}(D)$.

2.3. Norm-preserving operators. The group $G = \text{SU}(1,1)$ acts transitively on D by the linear transformation $g \cdot z = (\alpha z + \beta) / (\bar{\beta} z + \bar{\alpha})$ ($z \in D$ and $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G$) and the isotropy subgroup at $z=0$ is equal to K , so we have the identification $D = G/K$. Here we put $J(g, z) = \bar{\beta} z + \bar{\alpha}$, and define the operator $T_{p,r}(g)$ ($0 < p < \infty$, $r \in \mathbb{R}$ and $g \in G$) as follows.

$$T_{p,r}(g)F(z) = J(g^{-1}, z)^{-4(1+r)/p} F(g^{-1} \cdot z) \quad (z \in D). \quad (2.7)$$

Then it is easy to see that each operator $T_{p,r}(g)$ preserves $\| \cdot \|_{p,r}$ and $\| \cdot \|_{H^p}$ - "norms", and moreover, $(T_{p,r}, A_{p,r}(D))$ and $(T_{p,r}, H^p(D))$ are irreducible representations of G with respect to the topologies induced by the "norms".

§3. Discrete series of G . If $0 < p < \infty$ and $r \in \mathbb{R}$ satisfy the relation: $4(1+r)/p = 2n$, we put $T_n = T_{p,r}$. Then $(T_n, A_{2,n-1}(D))$ ($n \in \frac{1}{2}\mathbb{Z}$ and $n \geq 1$) are nothing but the holomorphic discrete series representation of G . Let $f_{\ell m}^n(g)$ ($g \in G$ and $\ell, m \in \mathbb{N}$) denote the normalized matrix coefficients of T_n defined by

$$f_{\ell m}^n(g) = [T_n(g)e_n^m, e_n^\ell]_{n-1} / \| [T_n(\cdot)e_n^m, e_n^\ell]_{n-1} \|_2, \quad (3.1)$$

where $[\cdot, \cdot]_r$ means the inner product of $A_{2,r}(D)$. Actually, by using hypergeo-

metric function we can give the explicit form of $f_{\ell m}^n$: for example, if $\ell \geq m$,

$$f_{\ell m}^n(g) = \frac{1}{2} \left(\frac{2n-1}{\pi} \right)^{1/2} \left(\frac{\Gamma(\ell+1)\Gamma(\ell+2n)}{\Gamma(m+1)\Gamma(m+2n)} \right)^{1/2} \frac{1}{\Gamma(\ell-m+1)} (1-r^2)^n \quad (3.2)$$

$$\times F(-m; 2n+\ell, \ell-m+1; r^2) r^{\ell-m} e^{-i\ell\theta} e^{-im\theta'} e^{-in(\theta+\theta')},$$

where $g = k_\theta a_t k_{\theta'}$, $\in KCL(A^+)K$ and $r = t/2$ (cf. [Sa]). Some basic properties of $f_{\ell m}^n$, which will be used in the following arguments, are summarized as follows.

Lemma 3.1. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$ and $\ell, m \in \mathbb{N}$.

$$(1) \quad \|f_{\ell m}^n\|_2 = 1.$$

$$(2) \quad f_{\ell m}^n(k_\theta g k_{\theta'}) = e^{-i(\ell+n)\theta} e^{-i(m+n)\theta'} f(g) \quad (k_\theta, k_{\theta'} \in K, g \in G).$$

$$(3) \quad T_n(g) e_n^m = 2 \left(\frac{\pi}{2n-1} \right)^{1/2} \sum_{\ell=0}^{\infty} f_{\ell m}^n(g) e_n^\ell.$$

$$(4) \quad \frac{1}{4\pi} \int_0^{4\pi} e^{i(u+n)\theta} f_{\ell m}^n(x k_\theta y) d\theta = 2 \left(\frac{\pi}{2n-1} \right)^{1/2} f_{\ell u}^n(x) f_{\ell m}^n(y) \quad (u \in \mathbb{N}, x, y \in G).$$

$$(5) \quad f_{\ell m}^n * f_{uv}^w = \delta_{nw} \delta_{mu} 2 \left(\frac{\pi}{2n-1} \right) f_{\ell v}^n \quad (w \in \frac{1}{2}\mathbb{Z}, w \geq 1 \text{ and } u, v \in \mathbb{N}).$$

$$(6) \quad f_{\ell m}^n \text{ is the eigenfunction of the Laplace-Beltrami operator of } G \text{ with the eigenvalue } 4n(n-1).$$

$$(7) \quad \text{Let } 1 \leq p \leq \infty \text{ and } (n, p) \neq (1, 1). \text{ Then } \|f_{\ell m}^n\|_p < \infty.$$

By the Plancherel formula for $L^2(G)$, each L^2 function f on G has the following decomposition:

$$f = f_p + \circ f, \quad \circ f = \sum_{\substack{n \in \frac{1}{2}\mathbb{Z}, |n| \geq 1 \\ \ell, m \in \mathbb{N}}} \hat{f}(n; \ell, m) f_{\ell m}^n, \quad (3.3)$$

where f_p consists of wave packets and $f_{\ell m}^n = \text{conj}(f_{\ell m}^{-n})$ for $n \leq -1$. Clearly this decomposition for $L^2(G) \cap L^p(G)$ ($1 \leq p \leq 2$) can be extended to $L^p(G)$. Then the next

lemma is an easy consequence from Lemma 3.1 (5), (6), (7) and (3.3).

Lemma 3.2. Let us suppose that f belongs to $L^p(G)$ ($1 \leq p \leq 2$). Then for $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$ and $\ell, m \in \mathbb{N}$ we have

$$(1) \quad f \star_P f_{\ell m}^n = 0.$$

$$(2) \quad f \star f_{\ell m}^n = c_n \sum_{u=0}^{\infty} \hat{f}(n; u, m) f_{u m}^n, \text{ where } c_n = 2 \left(\frac{\pi}{2n-1} \right)^{\frac{1}{2}}.$$

Therefore, if we put $P_m^n(f) = c_n^{-1} f \star f_{m m}^n$, P_m^n is the projection operator which maps f in $L^p(G)$ to the discrete part of f being of the form $\sum_{u=0}^{\infty} \hat{f}(n; u, m) f_{u m}^n$. One of the important properties of P_m^n is the following

Proposition 3.3. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$, $m \in \mathbb{N}$, $1 \leq p \leq 2$ and suppose that $(n, p) \neq (1, 1)$. Then $P_m^n(L^p(G))$ is contained in $L^p(G)$.

Proof. We shall use Lemma 3.1 (7). Let f be in $L^p(G)$. If $n > 1$ and $1 \leq p \leq 2$, then $\|P_m^n(f)\|_p = c_n^{-1} \|f \star f_{m m}^n\|_p \leq c_n^{-1} \|f_{m m}^n\|_1 \|f\|_p < \infty$, and if $n=1$ and $1 < p \leq 2$, $\|P_m^n(f)\|_p \leq C_p \|f\|_p < \infty$ by the Kunze-Stein phenomenon on G (see [C] and [CS]). Q.E.D.

§4. ϕ_m^n -transform. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$, $m \in \mathbb{N}$ and $1 \leq p \leq 2$. For an f in $L^p(G)$ we shall define $\phi_m^n(f)$ as follows.

$$\phi_m^n(f)(z) = c_n^{-1} T_n(\bar{f}) e_n^m(z) \quad (z \in D), \quad (4.1)$$

where $T_n(f)$ is the operator defined by $T_n(f) = \int_G f(g) T_n(g) dg$. By using Lemmas 3.1 and 3.2 we easily see the following

Lemma 4.1. $\phi_m^n(L^p(G)) = \phi_m^n(P_m^n(L^p(G)))$ and $\phi_m^n(\sum_{\ell} a_{\ell} f_{\ell m}^n) = \sum_{\ell} \bar{a}_{\ell} e_n^{\ell}$ for a finite sum Σ . In particular, ϕ_m^n is injective on $P_m^n(L^p(G))$.

Our aim is to give a characterization of $\phi_m^n(L^p(G))$ as a space of holomorphic functions on D . First we shall prove the independence of m for $\phi_m^n(L^p(G))$.

Theorem 4.2. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$, $m \in \mathbb{N}$ and $1 \leq p \leq 2$. Then $\phi_m^n(L^p(G)) = \phi_0^n(L^p(G))$.

Proof. Obviously, it is enough to prove that the correspondence of $f = \sum_{\ell=0}^{\infty} a_{\ell} f_{\ell m}^n$ to $\tilde{f} = \sum_{\ell=0}^{\infty} a_{\ell} f_{\ell 0}^n$ gives a bijection between $P_m^n(L^p(G))$ and $P_0^n(L^p(G))$. The case of $(n,p) \neq (1,1)$. Let us suppose that $f = \sum_{\ell=0}^{\infty} a_{\ell} f_{\ell m}^n$ belongs to $P_m^n(L^p(G))$. Then by Proposition 3.3 f also belongs to $L^p(G)$, and it follows from the same argument in the proof of Proposition 3.3 that $\|\tilde{f}\|_p = c \|f\|_p \leq c_{p,n} \|f\|_p < \infty$. Therefore, we see that $\tilde{f}$ belongs to $P_0^n(L^p(G))$. Clearly, this argument is reversible by noting that $f = c \tilde{f} * f_{0m}^n$. Thus the desired result follows from Lemma 4.1.

Before the proof of the case of $(n,p) = (1,1)$ we shall prove a lemma, which will play an important role in §5.

Lemma 4.3. For each $f_{\ell m}^1$ there exists an L^1 function $[f_{\ell m}^1]$ such that $f_{\ell m}^1 = [f_{\ell m}^1] * f_{mm}^1$.

Proof. For a fixed $\alpha > 0$ we put

$$[f_{\ell m}^1](g) = C_{\ell m}^{\alpha} |f_{00}^1(g)|^{\alpha} f_{\ell m}^1(g) \quad (g \in G),$$

where $C_{\ell m}^{\alpha}$ is the constant determined by

$$C_{\ell m}^{\alpha} c_1 \int_G |f_{00}^1(g)|^{\alpha} |f_{\ell m}^1(g)|^2 dg = 1. \quad (4.2)$$

Then it is easy to see that $[f_{\ell m}^1]$ belongs to $L^1(G)$ and moreover,

$$\begin{aligned} & [f_{\ell m}^1] * f_{mm}^1(x) \\ &= \int_G [f_{\ell m}^1](xy^{-1}) f_{mm}^1(y) dy \end{aligned}$$

$$\begin{aligned}
&= \int_G [f_{\ell m}^1](y^{-1}) \int_K e^{i(n+\ell)\theta} f_{mm}^1(yk_\theta x) dk dy \\
&= c_1 \int_G [f_{\ell m}^1](y^{-1}) \overline{f_{\ell m}^1}(y) dy f_{\ell m}^1(x) \\
&= c_1 C_{\ell m}^\alpha \int_G |f_{00}^1(y)|^\alpha |f_{\ell m}^1(y)|^2 dy f_{\ell m}^1(x) \\
&= f_{\ell m}^1(x).
\end{aligned}$$

Therefore, $[f_{\ell m}^1]$ satisfies the desired properties.

Q.E.D.

The case of $(n,p)=(1,1)$. Let us suppose that $f = \sum_{\ell=0}^{\infty} a_{\ell} f_{\ell m}^1$ belongs to $P_m^1(L^1(G))$. This means that there exists an L^1 function $[f]$ on G such that $f = [f] * f_{mm}^1$. Therefore, $\tilde{f} = c_1^{-1} f * f_{m0}^1 = [f] * f_{m0}^1$. Then by Lemma 4.3 we can choose an L^1 function $[f_{m0}^1]$ on G such that $f_{m0}^1 = [f_{m0}^1] * f_{00}^1$, and thus, $\tilde{f} = [f] * [f_{m0}^1] * f_{00}^1$. Here we note that $[f] * [f_{m0}^1]$ belongs to $L^1(G)$. Therefore, $\tilde{f}$ belongs to $P_0^1(L^1(G))$. Clearly, this argument is reversible as before, so we can obtain the desired result.

This completes the proof of the theorem.

Q.E.D.

§5. Characterization of $\Phi_0^n(L^p(G))$. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$ and $1 \leq p \leq 2$. By Theorem 4.2 our problem is reduced to the case of $m=0$. We shall give a characterization of $\Phi_0^n(L^p(G))$.

First we shall consider the case of $(n,p) \neq (1,1)$. Let us suppose that $f = \sum_{\ell=0}^{\infty} a_{\ell} f_{\ell 0}^n$ belongs to $P_0^n(L^p(G))$ and we put $F(z) = \Phi_0^n(f)(z) = \sum_{\ell=0}^{\infty} \bar{a}_{\ell} z^{\ell}$. Then we see that

$$\begin{aligned}
f(g) &= \frac{1}{2\pi} (1-r^2)^n \sum_{\ell=0}^{\infty} a_{\ell} B(\ell+1, 2n-1)^{-\frac{1}{2}} r^{\ell} e^{-i\ell\theta} e^{-in(\theta+\theta')} \\
&= \frac{1}{2\pi} (1-r^2)^n \overline{F(re^{i\theta})} e^{-in(\theta+\theta')},
\end{aligned} \tag{5.1}$$

where $g = k_{\theta} a_t k_{\theta^{-1}} \in KCL(A^+)K$ and $r = t/2$. Since f belongs to $L^p(G)$ by Proposition 3.3, this relation deduces that $\|f\|_p = c \|F\|_{p, \frac{1}{2}np-1} < \infty$. This means that

when $(n,p) \neq (1,1)$, f belongs to $P_0^n(L^p(G))$ if and only if $F = \phi_0^n(f)$ belongs to $A_{p, \frac{1}{2}np-1}(D)$, and ϕ_0^n preserves the norm up to a constant multiplication.

Next we shall consider the case of $(n,p)=(1,1)$. For a holomorphic function $F(z) = \sum_{\ell=0}^{\infty} a_{\ell} z^{\ell}$ on D the fractional derivative (resp. integral) of F of order $\alpha \geq 0$ is defined as follows.

$$F^{[\alpha]}(z) = \sum_{\ell=0}^{\infty} a_{\ell} \frac{\Gamma(\ell+1+\alpha)}{\Gamma(\ell+1)} z^{\ell} \quad (5.2)$$

(resp. $F_{[\alpha]}(z) = \sum_{\ell=0}^{\infty} a_{\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+1+\alpha)} z^{\ell}$).

By using this derivative we shall define

$$\|F\|_{H_0^1} = \lim_{\alpha \rightarrow 0} \alpha \|F^{[\alpha]}\|_{1, \frac{1}{2}\alpha - \frac{1}{2}}. \quad (5.3)$$

Then the space $H_0^1(D)$ is defined as the space of all holomorphic functions on D with finite $\|\cdot\|_{H_0^1}$ -norm.

Lemma 5.1. For each $g \in G$ $T_1(g)e_1^0(z)$ belongs to $H_0^1(D)$ and the norm is uniformly bounded on $g \in G$.

Proof. From the definition (2.7) it is easy to see that

$$z^{-1}(zT_1(g)e_1^0(z))^{[\alpha]} = \frac{r(\alpha+2)(1-r^2)e^{-i(\theta+\theta')}}{(1-re^{i\theta}z)^{\alpha+2}},$$

where $g = k_{\theta} a_t k_{\theta_1'} \in KCL(A^+)K$ and $r = t/2$. Then we have $\|(T_1(g)e_1^0)^{[\alpha]}\|_{1, \frac{1}{2}\alpha - \frac{1}{2}} \leq C\Gamma(\alpha+2)(1+\alpha^{-1})$, where C does not depend on $g \in G$. Thus the desired result is obtained. Q.E.D.

Now we shall give a characterization of $\phi_0^1(L^1(G))$. First let us suppose that f belongs to $L^1(G)$. Since $\phi_0^1(f)(z) = c_1^{-1}(T_1(\cdot)e_1^0(z), f)$, it follows from Lemma 5.1 that $\|\phi_0^1(f)\|_{H_0^1} \leq c_1^{-1}(\|T_1(\cdot)e_1^0\|_{H_0^1}, |f|) \leq c_1^{-1}c\|f\|_1 < \infty$.

This means that $\phi_0^1(f)$ belongs to $H_0^1(D)$ for all $f \in L^1(G)$.

Conversely let us suppose that $F(z) = \sum_{\ell=0}^{\infty} \bar{a}_{\ell} e_1^{\ell}(z)$ belongs to $H_0^1(D)$. Here we put $f(g) = \sum_{\ell=0}^{\infty} a_{\ell} f_{\ell 0}^1(g)$, and let $[f_{\ell 0}^1]$ ($\ell \in \mathbb{N}$) be the L^1 functions on G constructed in the proof of Lemma 4.3. Actually, for a fixed $\alpha > 0$ they are given by

$$[f_{\ell 0}^1](g) = c_{\ell 0}^{\alpha} |f_{00}^1(g)|^{\alpha} f_{\ell 0}^1(g) \quad (g \in G),$$

where

$$c_{\ell 0}^{\alpha} = \frac{2^{\alpha} \pi^{\frac{1}{2}(1+\alpha)} \Gamma(\ell+2+\alpha)}{\Gamma(\alpha+1) \Gamma(\ell+2)}$$

Here we put $[f] = \sum_{\ell=0}^{\infty} a_{\ell} [f_{\ell 0}^1]$. Then it follows from Lemmas 4.1 and 4.3 that $c_n \cdot \phi_0^1([f]) = \phi_0^1(f) = F$, and moreover,

$$\begin{aligned} \| [f] \|_1 &= \frac{2^{\alpha} \pi^{\frac{1}{2}(1+\alpha)}}{\Gamma(\alpha+1)} \| |f_{00}^1|^{\alpha} \sum_{\ell=0}^{\infty} a_{\ell} \frac{\Gamma(\ell+2+\alpha)}{\Gamma(\ell+2)} f_{\ell 0}^1 \|_1 \\ &= \frac{2\pi}{\Gamma(\alpha+1)} \| z^{-1}(zF)^{[\alpha]} \|_{1, \frac{1}{2}\alpha - \frac{1}{2}}. \end{aligned}$$

Obviously, this is finite, because F belongs to $H_0^1(D)$. Therefore, we see that F belongs to $\phi_0^1(L^1(G))$.

Summalizing the results we just obtained, we have the following

Theorem 5.2. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$ and $1 \leq p \leq 2$.

- (1) If $(n, p) \neq (1, 1)$, then $\phi_0^n(L^p(G)) = A_{p, \frac{1}{2}np-1}^n(D)$ and $\phi_0^n : P_0^n(L^p(G)) \rightarrow A_{p, \frac{1}{2}np-1}^n(D)$ is bijective and norm-preserving.
- (2) If $(n, p) = (1, 1)$, then $\phi_0^1(L^1(G)) = H_0^1(D)$ and $\phi_0^1 : P_0^1(L^1(G)) \rightarrow H_0^1(D)$ is bijective.

Remark 5.3. Noting the proof in the case of $(n, p) = (1, 1)$, we see that if

$\| F \|_{1, \frac{1}{2}\alpha - \frac{1}{2}} < \infty$ for an $\alpha > 0$, then $\| F \|_{1, \frac{1}{2}\alpha - \frac{1}{2}} < \infty$ for all $\alpha > 0$ and moreover $\| F \|_{H_0^1} < \infty$. Therefore, it is easy to see that for each $\alpha > 0$ $H_0^1(D)$ is

also defined as the space of all holomorphic functions F on D such that $F^{[\alpha]}$ belongs to $A_{1, \frac{1}{2}\alpha - \frac{1}{2}}(D)$.

Corollary 5.4. $L^1(G) \cap P_0^1(L^1(G)) = \{0\}$.

Proof. If $f \in L^1(G)$ belongs to $P_0^1(L^1(G))$, then $\|f\|_1 = 2\pi^{\frac{1}{2}} \|\phi_0^1(f)\|_{1, -\frac{1}{2}} < \infty$. Obviously, this means that $\phi_0^1(f) \equiv 0$ and thus $f \equiv 0$. Q.E.D.

§6. The space $H_0^1(D)$. In this section we shall state some basic properties of the space $H_0^1(D)$ introduced in the previous section.

Proposition 6.1.

(1) $H_0^1(D)$ is dense in $H^1(D)$.

(2) $H_0^1(D)$ is not contained in and does not contain $H^\infty(D)$. In particular $H_0^1(D) \subsetneq H^1(D)$.

Proof. Let f be in $L^1(G)$. Then $\|\phi_0^1(f)\|_{H^1} \leq (\|T_1(\cdot)e_1^0\|_{H^1}, |f|) \leq C\|f\|_1 < \infty$, and thus, $H_0^1(D) = \phi_0^1(L^1(G)) \subset H^1(D)$ (see Theorem 5.2 (2)). Here we note that $\phi_0^1(L^1(G))$ is G -invariant under the action of $T_1(g)$ ($g \in G$). Therefore, since $(T_1, H^1(D))$ is irreducible, $H_0^1(D)$ must be dense in $H^1(D)$. This proves (1).

Let F be an extremal function in $H^\infty(D)$. Then $|F'(z)| = (1-r^2)^{-1} \|F\|_\infty$ (cf. [D], p.144, Ex.7). Therefore, F' does not belong to $A_{1,0}(D)$. This means that F does not belong to $H_0^1(D)$ (see Remark 5.3), and thus, $H_0^1(D)$ does not contain $H^\infty(D)$. Let D be the space of all holomorphic functions F on D such that $\int_D |F'(z)|^2 dz < \infty$. Then by using Schwarz's inequality and Remark 5.3 we see that D is contained in $H_0^1(D)$. However, it is well-known that D is not contained in $H^\infty(D)$ (cf. [D], p.106, Ex.7). Thus $H_0^1(D)$ is not contained in $H^\infty(D)$. Moreover, since $H^\infty(D)$ is contained in $H^1(D)$, the last assertion is obvious.

Q.E.D.

Proposition 6.2. Let us suppose that F belongs to $H^1(D)$. Then for each $\alpha > 0$ $F_{[\alpha]}$ belongs to $H^1_0(D)$.

Proof. We easily see that $\| (F_{[\alpha]})^{[\alpha]} \|_{1, \frac{1}{2}\alpha - \frac{1}{2}} = \| F \|_{1, \frac{1}{2}\alpha - \frac{1}{2}} < \infty$ for each $\alpha > 0$ and F in $H^1(D)$. Then the assertion is clear from Remark 5.3. Q.E.D.

For $\mu \in \mathbb{R}$ and $a \in L^1(\partial D)$ we put

$$P_\mu(a)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)}{|1-re^{i(\theta-\phi)}|^{2(1+\mu)}} a(\phi) d\phi \quad (z=re^{i\theta}),$$

and let $H^1_{(\mu)}(D)$ be the space of all holomorphic functions on D being of the form $P_\mu(a)$ for an a in $L^1(\partial D)$. Then $H^1(D) = H^1_{(0)}(D)$ by [D], Corollary 2 in p. 34, and moreover we have the following

Proposition 6.3. For each $\mu < 0$ $H^1_{(\mu)}(D) \subset H^1_0(D) \subset H^1(D)$.

Proof. Let us suppose that F belongs to $H^1_{(\mu)}(D)$, that is, $F = P_\mu(a)$ for an a in $L^1(\partial D)$. Without loss of generality we may assume that $-\frac{1}{2} < \mu < 0$. By using the integral formula for the fractional derivative (cf. [K], (9.1)), $F^{[\epsilon]}$ can be written as

$$F^{[\epsilon]}(z) = \frac{1}{\Gamma(1-\epsilon)} \int_D F(\zeta) |\zeta|^{2\epsilon} (1-|\zeta|^2)^{-\epsilon} (1-\bar{\zeta}z)^{-2} d\zeta.$$

Now we shall take an $\epsilon > 0$ such that $0 < \epsilon < -2\mu$. Then by using Theorem A in [DRS] we see that

$$\begin{aligned} & \| F^{[\epsilon]} \|_{1, \frac{1}{2}\epsilon - \frac{1}{2}} \\ & \leq \frac{1}{\Gamma(1-\epsilon)} \frac{1}{2\pi} \| a \|_1 \sup_{0 \leq \phi < 2\pi} \int_D \frac{(1-|\zeta|^2)^{1-\epsilon}}{|1-\zeta e^{i\phi}|^{2(1+\mu)}} \int_D \frac{(1-|z|^2)^{\epsilon-1}}{|1-\bar{\zeta}z|^2} dz d\zeta \\ & \leq C \| a \|_1 \int_0^1 (1-r)^{1-\epsilon-2(1+\mu)} dr \int_0^1 (1-r)^{\epsilon-1} dr \\ & < \infty, \end{aligned}$$

where $|\zeta|=n$ and $|z|=r$. Then the desired result is obvious from Remark 5.3.

Q.E.D.

§7. Applications. We shall apply the characterization of $\phi_m^n(L^p(G))$ to the classical harmonic analysis on D . The following theorem, which is obtained in [DRS], Theorem 5, is an easy consequence of Theorem 5.2 and Remark 5.3.

Theorem 7.1. *Let $r > -\frac{1}{2}$ and $\beta \geq 0$. Then*

(1) *If F belongs to $A_{1,r}^{(D)}$, then $F^{[\beta]}$ belongs to $A_{1,r+\frac{1}{2}\beta}^{(D)}$.*

(2) *If F belongs to $A_{1,r+\frac{1}{2}\beta}^{(D)}$, then $F_{[\beta]}$ belongs to $A_{1,r}^{(D)}$.*

Moreover, if we replace $A_{1,-\frac{1}{2}}^{(D)} = \{0\}$ by the space $H_0^1(D)$, the assertions are also valid for the case of $r = -\frac{1}{2}$.

To the detail of the proof see [K].

Next we shall recall the explicit form of the matrix coefficients $f_{\ell m}^n$ (see (3.2)). Let $G_m(\alpha; \gamma; x)$ be the Jacobi polynomial with degree m . Then since $G_m(\ell-m+2n; \ell-m+1; 1) = (-1)^m \Gamma(2n+m) \Gamma(\ell-m+1) / \Gamma(2n) \Gamma(\ell+1)$, it is easy to see that $G_m(\ell-m+2n; \ell-m+1; r)$ has the expansion being of the form

$$\begin{aligned} & G_m(\ell-m+2n; \ell-m+1; r) \\ &= (-1)^m \frac{\Gamma(2n+m) \Gamma(\ell-m+1)}{\Gamma(2n) \Gamma(\ell+1)} r^m \\ &+ \frac{\Gamma(\ell-m+1)}{\Gamma(\ell)} (1-r) (Q_{m,m-1}^n(\ell) (1-r)^{m-1} + Q_{m,m-2}^n(\ell) (1-r)^{m-2} + \dots + Q_{m,0}^n(\ell)), \end{aligned}$$

where $Q_{m,k}^n$ is a polynomial of ℓ with degree k whose coefficients only depend on n, m and k . Therefore, since $F(-m; 2n+\ell, \ell-m+1; r) = G_m(\ell-m+2n; \ell-m+1; r)$, $f_{\ell m}^n(g)$

($\ell \geq m$, $g = k_\theta a_t k_\theta$, and $r = tht/2$) has the expansion:

$$f_{\ell m}^n(g) = C_m^n f_{\ell 0}^n(g) r^{-m} \{ r^{2m} + (1-r^2) p_{m,m}^n(\ell) + \dots + (1-r^2) p_{n,1}^n(\ell) \} e^{-m\theta}, \quad (7.1)$$

where $C_m^n = (-1)^m (\Gamma(m+2n)/\Gamma(m+1)\Gamma(2n))^{\frac{1}{2}}$ and $p_{m,k}^n(\ell) = (-1)^m \frac{\Gamma(2n)}{\Gamma(2n+m)} \ell Q_{m,k}^n(\ell)$. By using these polynomials $p_{m,k}^n$, we shall define a differential operator $D_{m,k}^n$ as follows. For a holomorphic function $F(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$

$$D_{m,k}^n F(z) = \sum_{\ell=0}^{\infty} p_{m,k}^n(\ell) a_\ell z^\ell. \quad (7.2)$$

Moreover, we put

$$D_m^n F(z) = C_m^n \{ F_{(m)} |z|^m + (1-|z|^2)^m D_{m,m}^n F_{(m)}(z) |z|^{-m} + (1-|z|^2)^{m-1} D_{m,m-1}^n F_{(m)}(z) |z|^{-m} + \dots + (1-|z|^2)^{-1} D_{m,1}^n F_{(m)}(z) |z|^{-m} \} \quad (7.3)$$

where $F_{(m)}(z) = \sum_{\ell=m}^{\infty} a_\ell z^\ell$.

Now we shall recall Proposition 4.2 and its proof. Then we see that $F(z) = \sum_{\ell=0}^{\infty} a_\ell e_n^\ell(z)$ belongs to $\Phi_0^n(L^p(G))$ if and only if $\sum_{\ell=0}^{\infty} a_\ell f_{\ell m}^n$ belongs to $P_m^n(L^p(G))$, and thus by (7.1), (7.3), if and only if $\|D_m^n F\|_{p, \frac{1}{2}np-1} < \infty$ when $(n,p) \neq (1,1)$, and $\|D_m^n F^{[\alpha]}\|_{1, \frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for an $\alpha > 0$ when $(n,p) = (1,1)$. Here let $A_{p, \frac{1}{2}np-1}(D)_m$ (resp. $H_0^1(D)_m$) be the space of all holomorphic functions F on D such that $\|D_m^n F\|_{p, \frac{1}{2}np-1} < \infty$ (resp. $\|D_m^n F^{[\alpha]}\|_{1, \frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for an $\alpha > 0$). Then we have the following

Theorem 7.2. Let $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$ and $1 \leq p \leq 2$.

(1) If $(n,p) \neq (1,1)$, then $A_{p, \frac{1}{2}np-1}(D)_m = A_{p, \frac{1}{2}np-1}(D)$ for $m \in \mathbb{N}$.

(2) If $(n,p) = (1,1)$, then $H_0^1(D)_m = H_0^1(D)$ for $m \in \mathbb{N}$.

Corollary 7.3. *Let us suppose that $(n,p) \neq (1,1)$. Then if F belongs to $A_{p, \frac{1}{2}np-1}^{(D)}$, then F' belongs to $A_{p, \frac{1}{2}(n+1)p-1}^{(D)}$.*

Proof. This is an easy consequence from the case of $m=1$ in Theorem 7.2.

Q.E.D.

References

- [C] Cowling, M.: The Kunze-Stein phenomenon, Ann. of Math., 107 (1978), 209-234.
- [CR] Coifman, R.R. and Rochberg, R.: Representation theorems for holomorphic and harmonic functions in L^p , Asterisque 77 (1980), 12-66.
- [CS] Clerc, J.L. and Stein E.M.: L^p -multipliers for non-compact symmetric spaces, Proc. Nat. Acad. Sci. U.S.A., 71 (1974), 3911-3912.
- [D] Duren, P.L.: Theory of H^p Spaces, Academic Press, New York, 1970.
- [DRS] Duren, P.L., Romberg, B.W. and Schields, A.L.: Linear functionals on H^p spaces with $0 < p < 1$, Reine Angew. Math., 238 (1969), 32-60.
- [K] Kawazoe, T.: Fourier coefficients of L^p functions on $SU(1,1)$, Keio Univ. Research Report, 7 (1985).
- [Sa] Sally, P.J.: Analytic continuation of irreducible unitary representations of the universal covering group of $SL(2,R)$, Memoirs of A.M.S., 69 (1967).
- [Su] Sugiura, M.: Unitary Representations and Harmonic Analysis, Wiley, New York, 1975.