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Invariant Holonomic Systems and Invariant Tempered Distributions on Regular Prehomogeneous Vector Spaces

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Introduction.

The purpose of this expository paper is to explain a method to determine all the invariant tempered distributions or relatively invariant tempered distributions on a real affine space X under the action of a linear algebraic subgroup G in $GL(X)$ when X decomposes into finitely many G-orbits.

For example, consider the space $X:= M_n(\mathbb{R})$ of all the $n\times n$ matrices over the real number field. The group $G:=GL_n(\mathbb{R})^{+}\times SL_n(\mathbb{R})$ acts on X by $x \longmapsto g\cdot x=g_1\cdot x\cdot {}^t g_2$ with $x\in X$ and $g=(g_1,g_2)\in G$ and X decomposes into finitely many orbits by the action of G. We put $\chi(g):=(\det(g))$. We call a tempered distribution $f(x)\in\mathcal{S}'(X)$ a $\chi(g)^s$-*invariant tempered distribution* if it satisfies $f(g\cdot x)=\chi(g)^s f(x)$ for all $g\in G$. Whenever s is not a negative integer, it is written as a linear combination of the two kinds of complex powers of $\det(x)$: $|\det(x)|^s$ and $\mathrm{sgn}(\det(x))|\det(x)|^s$ with $s\in\mathbb{C}$. Here $\mathrm{sgn}(*)$ means the signature of the value $(*)$. The above complex powers are defined firstly in the domain that the real part $\mathrm{Re}(s)$ is sufficiently large and secondly continued to the whole space $\mathbb{C}$ by meromporphic continuation. But when s ia a negative integer, there exists a $\chi(g)^s$-invariant tempered distribution whose support is contained in $S:=\{x\in X;\ \det(x)=0\}$ and it seems that it is not written as a linear combination of the above complex powers. However we may in fact prove that any $\chi(g)^{\alpha}$-invariant tempered distribution is written as:

$$a_1(s)\cdot|\det(x)|^s + a_2(s)\cdot\mathrm{sgn}(\det(x))|\det(x)|^s|_{s=\alpha} \quad ,$$

where $a_1(s)$ and $a_2(s)$ are holomorphic functions near $s=\alpha$. At the same time, we may show that the dimension of $\chi(g)^{\alpha}$-invariant tempered distributions. We put $G^1:=\{g\in G;\chi(g)=1\}$. We call a tempered distribution on X whose support is contained in the singular set S a *singular invariant distribution*. Then any singular invariant tempered distribution is written as a finite linear combination of the Laurant coefficients of negative order appeared in poles of the above two complex powers.

It is expected that the above situation would be valid in a more general situation and in fact the author recognized the above phenomena in a wider class of pairs (G,X). In the following the author will explain how to apply the theory of holonomic systems to this problem.

This problem seems to be firstly proposed by Weil [We] in an attempt to extend Iwasawa-Tate Theory on zeta functions. In [We], Weil requested to solve the determination problem of relatively invariant tempered distribution for all the pairs (G,X) defined over all the local fields while our method is well worked only in the case when the pair (G,X) is defined over the real field. After appearing Weil's paper, some interesting papers have been published, for example, Raïs,M [Ra], Rubenthaler [Ru] , Ricci-Stein [Ric-St] and so on. They dealt with related problems and obtained some interesting results by using Fourier analysis or classical functional analysis. But the author thinks that this problem is suitable for an application of the theory of holonomic systems and it enable us to solve the problem in a wider class at least when we consider the case that (G,X) is defined over the real field $\mathbb{R}$. For a pair (G,X) defined over other local fields (especially with discrete valuations), the author has no idea like an application of microlocal analysis.

1. Formulation of the problem.

Let X be a finite dimensional vector space defined over the real number field $\mathbb{R}$ and let G be a connected linear algebraic subgroup of $GL(X)$. We denote by ρ the inclusion map $G \hookrightarrow GL(X)$. We denote by $G_{\mathbb{C}}$ the complexification of G. The complexification $G_{\mathbb{C}}$ is a complex linear algebraic subgroup of $GL(X_{\mathbb{C}})$ where $X_{\mathbb{C}}$ is the complex linear vector space which is a complexification of X. We suppose that:

(1.1) $X_{\mathbb{C}}$ decomposes into a finite number of $G_{\mathbb{C}}$-orbits.

The condition (1.1) implies that $X_{\mathbb{C}}$ containes one open dense $G_{\mathbb{C}}$-orbit , which means that $(G_{\mathbb{C}},X_{\mathbb{C}})$ forms a prehomogeneous vector space and that the real vector space X decomposes into a finite number of orbits by the action of G. The group G (resp. $G_{\mathbb{C}}$) acts on the dual real vector space X^* (resp. the dual complex vector space

$X_{\mathbb{C}}{}^{*}$) through the contragredient representation ρ^{*}.

Let ν be a real analytic representation of G to the complex finite dimensional vector space V. We say that a V-valued hyperfunction u(x) a *ν-invariant hyperfunction* if:

$$(1.2) \quad u(\rho(g)x) = \nu(g)u(x), \qquad \text{for all} \quad g \in G.$$

Our problem is the following.

Main Problem

Determine all the ν-invariant hyperfunction on X.

When ν is a rational representation and if the solution u(x) is restricted to be a rational function, the problem is reduced to the realization problem of G in the space of rational functions on X and it may be solved by an algebraic method. However if ν is actually real analytic and u(x) is allowed to be a hyperfunction or a tempered distribution on X, we have to take functions whose support is contained in the comlement of the open orbits, which we call *singular distributions*, into the consideration. Such functions are not suitable for algebraic manipulation. In order to solve the main problem in hyperfunction category, we apply the theory of holonomic system to our problem. That is to say, we rewrite the condition for u(x) in (1.2) by taking an infinitesimal action of G and obtain a kind of linear differential equations called a *holonomic system*. We denote by $\mathfrak{g}$ the real Lie algebra of G. Let $d\nu$ be the infinitesimal representation of ν. Consider the following syatem of linear differential equations:

$$(1.3)\quad \mathfrak{M}_{(d\nu)} : (\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\rangle - d\nu(A))u(x) = 0 \text{ , for all } A\in\mathfrak{g}.$$

Here $\langle x,y\rangle$ is the canaonical bilinear form on $(x,y)\in X\times X^*$; $d\rho$ is the infinitesimal representation of ρ; $u(x)$ is the unknown function of the system of linear differential equation $\mathfrak{M}_{(d\nu)}$, which may be taken to be a V-valued hyperfunction on X. From the assumption that G is connected, the condition for $u(x)$ by the equations (1.3) is equivalent to the limitation on $u(x)$ by the group action in (1.2). Namely $u(x)$ is ν-invariant if and only if $u(x)$ is a solution to the system of linear differential equations $\mathfrak{M}_{(d\nu)}$. Thus the determination problem of ν-invariant hyperfunctions is reduced to the construction of hyperfunction solutions of the system $\mathfrak{M}_{(d\nu)}$. The system $\mathfrak{M}_{(d\nu)}$ has some good properties.

Consider the folllowing system of linear differential equation in the complex domain:

$$(1.3)' \qquad (\langle d\rho(A)\cdot z, \frac{\partial}{\partial z}\rangle - d\nu(A))u(z) = 0 \text{ , for all } A\in\mathfrak{g}_{\mathbb{C}}.$$

where (z) is the complex coordinate in $X_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ is the Lie algebra $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\oplus\sqrt{-1}\mathfrak{g}$. Then $\mathfrak{M}_{(d\nu)}$ is a real form of the system (1.3). We use tha same notation $\mathfrak{M}_{(d\nu)}$ for the system $(1.3)'$. Then the system $\mathfrak{M}_{(d\nu)}$ is a holonomic system on $X_{\mathbb{C}}$: a system of linear differential equation whose characteristic variety is a Lagrangian subvariety in $T^*X_{\mathbb{C}}=X_{\mathbb{C}}\times X^*_{\mathbb{C}}$. For the exact definition of a holonomic system, see Kashiwara [Ka]. A remarkable property of a holonomic system is that the hyperfunction solution space of a holonomic system is finite dimensional (Theorem 5.1.1 in [Ka]). More precisely, we have the folllowing proposition:

Proposition 1. 1) *$\mathfrak{M}_{(d\nu)}$ is a completely regular holonomic system on* X.

2) *The characteristic variety* $\mathrm{ch}(\mathfrak{M}_{(d\nu)})$ *is given by:*

(1.4) $\mathrm{ch}(\mathfrak{M}_{(d\nu)}) = \bigcup_i T^*_{X_{i\mathbb{C}}}X_{\mathbb{C}}$

where $\amalg_i X_{i\mathbb{C}}$ *is the* $G_{\mathbb{C}}$*-orbital decomposition of* X *and* $T^*_{X_{i\mathbb{C}}}X_{\mathbb{C}}$ *stands for the conormal bundle of* $X_{i\mathbb{C}}$.

Let $u(x)$ be a hyperfunction solution of the system $\mathfrak{M}_{(d\nu)}$. We denote by sp the spectral map from the sheaf $\mathcal{B}_X$ of the hyperfunctions on X to the sheaf $\pi_*(\mathcal{C}_X)$ where $\mathcal{C}_X$ is the sheaf of microfunctions on T^*X and π is the projection map from T^*X to X. Then sp gives an isomorphism $\mathcal{B}_X \simeq \pi_*(\mathcal{C}_X)$ and an exact sequence: $0 \longrightarrow \mathcal{A}_X \longrightarrow \mathcal{B}_X \longrightarrow \pi_*(\mathcal{C}_X|_{T^*X-X}) \longrightarrow 0$ (Sato's fundamental exact sequence), where $\mathcal{A}_X$ is the sheaf of real analytic functions on X. The support of $sp\circ u(x)$ in T^*X, which we denote by $\mathrm{supp}(sp\circ u(x))$, is called the support of $u(x)$ as a microfunction which is a closed real analytic subset in T^*X. Then we have:

Proposition 2. ([Mr4])

1) $\mathrm{supp}(sp\circ u(x)) \subset \mathrm{ch}(\mathfrak{M}_{(d\nu)})\cap T^*X =: \mathrm{ch}(\mathfrak{M}_{(d\nu)})_{\mathbb{R}}$.

2) *For two solutions* $u_1(x)$ *and* $u_2(x)$ *to* $\mathfrak{M}_{(d\nu)}$, *if* $sp\circ u_1(x)$ *and* $sp\circ u_2(x)$ *coincide with each other on* $\mathrm{ch}(\mathfrak{M}_{(d\nu)})_{\mathrm{reg}\mathbb{R}}$, *then* $u_1(x)=u_2(x)$

as hyperfunctions on X. *Here* $ch(\mathfrak{M}_{(d\nu)})_{reg\mathbb{R}}$ *is the real locus of the set of regular points of the complex analytic set* $ch(\mathfrak{M}_{(d\nu)})$.

From this proposition we have to study what solution is allowed to be exists only on the subset $ch(\mathfrak{M}_{(d\nu)})_{reg\mathbb{R}}$ but not on the whole $ch(\mathfrak{M}_{(d\nu)})_{\mathbb{R}}$. Since $ch(\mathfrak{M})_{reg\mathbb{R}}$ is a non-singular subvariety, we may investigate the behavior of the solutions more easily than the whole domain $ch(\mathfrak{M}_{(d\nu)})_{\mathbb{R}}$.

2. Some theorems.

Let (G,X) be a pair satisfying the condition (1.1) and let $(G_{\mathbb{C}},X_{\mathbb{C}})$ be the complexification of (G,X). We put $S_{\mathbb{C}}$ the complement of the open $G_{\mathbb{C}}$-orbit in $X_{\mathbb{C}}$ and let S be its real locus: $S=S_{\mathbb{C}}\cap X$. We suppose that:

(2.1) There exists an irreducible homogeneous polynomial $P(x)$ on $X_{\mathbb{C}}$ such that:

i) $S_{\mathbb{C}}=\{x\in X_{\mathbb{C}};P(x)=0\}$.

ii) The Hessian of $P(x)$: $\det(\partial P/\partial x_i\partial x_j)$ does not identically vanish on X.

iii) $P(x)$ is a polynomial with real coefficients when we restrict $P(x)$ on X.

The condition (2.1) is a sufficient condition for that $(G_{\mathbb{C}},X_{\mathbb{C}})$ is a regular prehomogeneous vector space. $P(x)$ is a relatively invariant polynomial on $X_{\mathbb{C}}$. There exists a character $\chi(g)$ of $G_{\mathbb{C}}$ such that

$P(g\cdot x)=\chi(g)P(x)$. Since $P(x)$ is with real coefficients, $\chi(g)$ is a real-valued character on G. Let $O_1\cup\cdots\cup O_\ell$ be the connected component decomposition of the set $X-S$. Then each O_i $(i=1,\cdots,\ell)$ is a G-orbit. We put:

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & x\in O_i \\ 0 & x\notin O_i \end{cases}$$

where s is a complex number. If the real part $\mathrm{Re}(s)$ is sufficiently large, then $|P(x)|_i^s$ is a continuous function holomorphically depending on s and it can be continued to the whole complex plane as a tempered distribution with a meromorphic parameter $s\in\mathbb{C}$. Apparently, for a fixed $\lambda\in\mathbb{C}$, $|P(x)|_i^\lambda$ is a $\chi(g)^\lambda$-invariant tempered distribution and hence it is a solution of the holonomic system $\mathfrak{M}_{(\lambda d\chi)}$. Moreover let $a_i(s)$ $(i=1,\cdots,\ell)$ be meromorphic functions defined near $s=\lambda$ and suppose that $v_s(x)=\sum_{i=1}^{\ell}a_i(s)\cdot|P(x)|_i^s$ is a tempered distribution with a holomorphic parameter s near $s=\lambda$. Then the tempered distribution $v(x)=v_s(x)|_{s=\lambda}$ is a χ^λ-invariant tempered distribution. We say that $v(x)$ is a *linear combination* of $|P(x)|_i^s$ at $s=\lambda$. Let λ_k $(k=0,1,2,\cdots)$ be a point in $\mathbb{C}$ where $|P(x)|_i^s$ has a pole. We have the Laurent expansion at $s=\lambda_k$:

$$(2.2)\qquad |P(x)|_i^s = \sum_{-j_k\le j<+\infty} T_i{}^j{}_k(x)\cdot(s-\lambda_k)^j \quad ,$$

where j_k is the order of the pole at $s=\lambda_k$. We put $G^1:=\{g\in G;\chi(g)=1\}$. Then it is easy to see that $T_i{}^j{}_k(x)$ is a G^1-invariant tempered distribution whose support is contained in the set S if $j<0$. For a tempered distribution $T(x)$ on X, we call $T(x)$ a *singular tempered distribution* if the support of $T(x)$ is contained in the set S. That is to say, each $T_i{}^j{}_k(f)$ is a G^1-invariant singular tempered distribution.

Theorem 3. ([Mr2]) *We suppose that* S *decomposes into a finite number of* G^1*-orbits. The following conditions* (A) *and* (B) *are equivalent under the condition* (2.1):

(A) *Any* G^1*-invariant singular tempered distribution is obtained as a finite linear combination of* $T_i{}^j{}_k(x)$ *in* (2.2) *with* $1\leq i\leq \ell, k=1,2,\cdots,$ *and* $j_k\leq j<0$.

(B) *Let* $\lambda\in\mathbb{C}$. *Any relatively invariant tempered distribution* $T(x)$ *corresponding to the character* $\chi(g)^\lambda$ *is obtained as a linear combination of* $|P(x)|_i^s$ $(i=1,\cdots,\ell)$ *at* $s=\lambda$.

Given a pair (G,X) satisfying the condition (2.1) and the assumption of Theorem 3, we can actually prove that the condition (B) is valid by case-by-case calculations. The principal tool is microlocal analysis explained particularly in §1 and §2. For example we have:

Theorem 4. ([Mr3]) *For almost all real forms (listed below) of prehomogeneous vector space of commutative parabolic type (for the definition of prehomogeneous vector spaces of commutative parabolic type, see* [Mu-Ru-Sc] *and* [Ru2]*), the condition (B) in Theorem 3 is valid.*

The list of the real forms of prehomogeneous vector spaces of commutative parabolic type.

1) The type C_m $(m=1,2,\ldots)$.

i) $(G,X) = (GL_m(\mathbb{R})^+, Sym_m(\mathbb{R}))$.

Here, $Sym_m(\mathbb{R})$ stands for the space of m×m symmetric matrices over $\mathbb{R}$.

2) The type A_k $(k=2m+1, m=1,2,\ldots)$.

i) $(G,X) = (GL_1(\mathbb{R})^+ \times SL_m(\mathbb{C}), Her_m(\mathbb{C}))$.

Here $Her_m(\mathbb{C})$ is the space of m×m complex Hermitian matrices.

ii) $(G,X) = (GL_m(\mathbb{R})^+ \times SL_m(\mathbb{R}), M_m(\mathbb{R}))$.

Here $M_m(\mathbb{R})$ is the space of m×m real matrices.

3) The type $D_{2m,2}$ $(m=1,2,\ldots)$.

i) $(G,X) = (GL_1(\mathbb{R})^+ \times SL_m(\mathbb{H}), Her_m(\mathbb{H}))$.

Here $\mathbb{H}$ stands for the quaternion field over $\mathbb{R}$ and $Her_m(\mathbb{H})$ is the space of m×m quaternion Hermitian matrices.

ii) $(G,X) = (GL_{2m}(\mathbb{R})^+, Alt_{2m}(\mathbb{R}))$.

Here $Alt_{2m}(\mathbb{R})$ stands for the space of 2m×2m alternative matrices over $\mathbb{R}$.

4) The type E_7.

i) $(G,X) = (GL_1(\mathbb{R})^+ \times E_6{}^d, Her_3(\mathfrak{C}_{\mathbb{R}}{}^d))$.

Here, $\mathfrak{C}_{\mathbb{R}}{}^d$ is the space of division Cayley number field over $\mathbb{R}$ and $Her_3(\mathfrak{C}_{\mathbb{R}}{}^d)$ means the space of 3×3 Hermitian matrices over $\mathfrak{C}_{\mathbb{R}}{}^d$. The group $E_6{}^d$ is the subgroup of

$GL^+(Her_3(\mathbb{C}_{\mathbb{R}}{}^d))$ consisting of the elements which remain $P(x)=\det(x)$ invariant.

ii) $(G,X) = (GL_1(\mathbb{R})^+ \times E_6{}^s, Her_3(\mathbb{C}_{\mathbb{R}}{}^s))$.

Here, $\mathbb{C}_{\mathbb{R}}{}^s$ is the space of split Cayley number algebra over $\mathbb{R}$ and $Her_3(\mathbb{C}_{\mathbb{R}}{}^s)$ means the space of 3×3 Hermitian matrices over $\mathbb{C}_{\mathbb{R}}{}^s$. The group $E_6{}^s$ is the subgroup of $GL^+(Her_3(\mathbb{C}_{\mathbb{R}}{}^s))$ consisting of the elements which remain $P(x)=\det(x)$ invariant.

5) The type B_k $(m=2k+1)$ and $D_{k+1,1}$ $(m=2(k+1))$ with $k=1,2,..$

i) $(G,X) = (GL_1(\mathbb{R})^+ \times SO(p,q;\mathbb{R}), \mathbb{R}^m)$, $(p,q>0$ and $p+q=m)$.

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