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Representations of Weyl groups and Hecke rings on virtual character modules of a semisimple Lie group

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§0 <u>Introduction.</u> Let G be a semisimple Lie group. The irreducible admissible representations of G was classified by Langlands. However, his method is inductive one and parameters of irreducible representations are not so available that one can proceed to harmonic analysis on them. So we change the point of view, and want not to classify each representation but to classify representations into some classes. We think that useful classification theory can be developped using representations of Weyl groups on virtual character modules. Weyl group(or Hecke ring) representations will classify irreducible representations into some classes, which have good invariants such as Gel'fand -Kirillov dimension, Borho-Jantzen-Duflo's τ-invariant and so on.

Here, we define representations of Weyl groups and Hecke rings on virtual character modules of G and state some results which are directly deduced from the definitions. Classification theory using them is to be mentioned in the near future.

§1 <u>Preliminaries and notations.</u> From now on, G is a connected semisimple Lie group with finite centre. We also assume G to be acceptable for technical reasons. Let $\mathfrak{g}$ be a Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We denote by $U(\mathfrak{g}_{\mathbb{C}})$ the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Take an algebra homomorphism χ of the centre $\underline{Z}$ of $U(\underline{g}_{\mathbb{C}})$ into the complex number field $\mathbb{C}$. We put $\mathrm{Mod}(\chi)=\{[M] \mid M$ is an irreducible admissible $(\underline{g}_{\mathbb{C}},K)$-module on which the centre $\underline{Z}$ acts as scalar operators through the morphism $\chi\}$, where $[M]$ denotes the equivalence class of M and K is a maximal compact subgroup of G . We know $\#\mathrm{Mod}(\chi)$ is finite. We denote by $\Theta(M)$ $(M \in \mathrm{Mod}(\chi))$ the grobal character of M which is an invariant eigendistribution(IED) on G by definition. Put

$$V(\chi)=\bigoplus_{[M]\in \mathrm{Mod}(\chi)} \mathbb{C}\Theta(M) \quad .$$

We call $V(\chi)$ a virtual character module with infinitesimal character χ . One can prove that $V(\chi)$ is the space of constant coefficient IEDs on G which have the eigenvalue χ (see [6]).

Let $\mathrm{Car}(G)$ be the set of all the cojugacy classes of Cartan subgroups of G . We fix $[H]\in \mathrm{Car}(G)$ for a while ($[H]$ is the cojugacy class of H). Through the Harish-Chandra map, $\underline{Z}$ is isomorphic to $U(\underline{h}_{\mathbb{C}})^W$ as C-algebras, where $W=W(\underline{g}_{\mathbb{C}},\underline{h}_{\mathbb{C}})$ is the Weyl group of $(\underline{g}_{\mathbb{C}},\underline{h}_{\mathbb{C}})$. On the other hand, since $U(\underline{h}_{\mathbb{C}})$ is a polynomial ring, an algebra homomorphism of $U(\underline{h}_{\mathbb{C}})$ into $\mathbb{C}$ is determined by $\lambda \in \underline{h}_{\mathbb{C}}^*$. We denote by χ_λ the corresponding homomorphism in $\mathrm{Hom}_{alg}(\underline{Z},\mathbb{C})$.

$$\begin{array}{ccccc} \mathrm{Hom}_{alg}(\underline{Z},\mathbb{C}) & \xrightarrow{\sim} & \mathrm{Hom}_{alg}(U(\underline{h}_{\mathbb{C}})^W,\mathbb{C}) & \xrightarrow{\sim} & \underline{h}_{\mathbb{C}}^*/W \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ \chi_\lambda & \longleftarrow & \longrightarrow & & \lambda \end{array}$$

Remark that $\chi_\lambda = \chi_{w\lambda}$ for any $w \in W$. In the following, we also fix λ and write $\chi = \chi_\lambda$. Define a subgroup $W_H(\lambda)$ of W as follows.

<u>Definition 1.1.</u> Let H_0 be the identity component of H. We put

$$\widetilde{W}_H(\lambda) = \left\{ w \in W \;\middle|\; \begin{array}{l} \exp x \longrightarrow \exp w\lambda(x) \quad (x \in \underline{h}) \text{ defines a well-} \\ \text{defined character on } H_0 . \end{array} \right\}$$

Then $W_H(\lambda)$ is defined to be the largest subgroup of W which leaves $\widetilde{W}_H(\lambda)$ invariant under the right multiplication. We call $W_H(\lambda)$ <u>integral Weyl group</u> of λ and H.

For $w \in \widetilde{W}_H(\lambda)$, define a character on H_0 by $\xi_{w\lambda}(\exp x) = \exp w\lambda(x)$ $(x \in \underline{h})$. For a root α, we also define the similar character on the whole H by $\mathrm{Ad}(h)X_\alpha = \xi_\alpha(h)X_\alpha$ $(h \in H)$, where X_α is a non-zero root vector for α. Let us define some more functions on H.

Let $\Delta = \Delta(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$ be the root system of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$ and choose a positive system Δ^+. Put $\Delta_{\mathbb{R}} = \{\alpha \in \Delta \mid \alpha(\underline{h}) \subset \mathbb{R}\}$ and $\Delta_{i\mathbb{R}} = \{\alpha \in \Delta \mid \alpha(\underline{h}) \subset \sqrt{-1}\,\mathbb{R}\}$. We call elements of $\Delta_{\mathbb{R}}$ real roots and those of $\Delta_{i\mathbb{R}}$ imaginary roots. Let $\rho \in \underline{h}^*_{\mathbb{C}}$ be half the sum of positive roots. Then we define the Weyl denominator by

$$D(h) = \xi_\rho(h) \prod_{\alpha\in\Delta^+} (1-\xi_\alpha(h)^{-1}) \quad (h\in H) .$$

We also define a function on H which takes values in $\{\pm 1\}$:

$$\varepsilon_{\mathbb{R}}(h) = \prod_{\alpha\in\Delta^+_{\mathbb{R}}} \mathrm{sgn}\ (1-\xi_\alpha(h)^{-1}) \quad (h\in H) .$$

Put $W(G;H)=N_G(H)/Z_G(H)$. Then a function $\varepsilon(\ ;\ )$ on $H\times W(G;H)$ is defined by the following equation: $\varepsilon_{\mathbb{R}}D(sh)=\varepsilon(h;s)\varepsilon_{\mathbb{R}}D(h)$ $(h\in H,$ $s\in W(G;H))$.

§2. Representations of Weyl groups. In the following, we assume that $\lambda\in \underline{h}^*_{\mathbb{C}}$ satisfies the following assumption.

Assumption 2.1. Let $H_0, H_1, \ldots, H_k$ be a complete system of representatives of $W(G;H)$-conjugacy classes of connected components of H . Then there exists $a_i\in H_i$ $(0\leq i\leq k)$ such that

(*) $\quad \xi_{t\lambda}(a_i{}^{-1}sa_i)=1 \quad (t\in \widetilde{W}_H(\lambda),\ s\in W(G;H_i))$,

where $W(G;H_i)=N_G(H_i)/Z_G(H_i)$.

Remark1. Since a_i and sa_i belongs to H_i , $a_i{}^{-1}sa_i$ belongs to H_0 and (*) has meaning. For i=0, we always put $a_0=e$ (identity element of G).

Remark2. Assumption 2.1 is satisfied for any λ in case that $G= SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $SO_0(p,q)$ $(p+q=2n)$. We cannot find a group which does not satisfy the assumption yet. See also Lemma1.6 in [7].

In the sequel, we assume all the elements in $\underline{h}_{\mathbb{C}}^*$ which appears in this report satisfies Assumption 2.1 for common $\{a_i\}$.

Now we define a class of analytic functions $\underline{B}(H;\lambda)$ on H after T.Hirai[2,3]. Define $\{H_i\}$ as in Assumption 2.1. For $0 \leqq i \leqq k$ and $t \in \widetilde{W}_H(\lambda)$, we construct an analytic function $\zeta(i,t;*)$ as follows. On H_i , it is given by the equation:

$$\zeta(i,t;a_i \exp x) = \sum_{s \in W(G;H_i)} \varepsilon(a_i;s) \exp st\lambda(x) \quad (x \in \underline{h}) .$$

On W(G;H)-orbit of H_i , it is given by $\zeta(i,t;wh) = \varepsilon(h;w)\zeta(i,t;h)$ $(h \in H_i,\ w \in W(G;H))$. Finally, outside the W(G;H)-orbit of H_i , we define $\zeta(i,t;h)=0$. We put

$$\underline{B}(H;\lambda) = \langle \zeta(i,t;*) \mid 0 \leqq i \leqq k,\ t \in \widetilde{W}_H(\lambda) \rangle / \mathbb{C} ,$$

$$\underline{B}^i(H;\lambda) = \langle \zeta(i,t;*) \mid t \in \widetilde{W}_H(\lambda) \rangle / \mathbb{C} ,$$

and call elements of $\underline{B}(H;\lambda)$ <u>ε-symmetric functions of constant coefficients</u> with eigenvalue λ .

<u>Theorem 2.2(T.Hirai[3];[6]).</u> There exists an isomorphism of the vector spaces:

$$\bigoplus_{[H] \in Car(G)} \underline{B}(H;\lambda) \xrightarrow[T]{\sim} V(\chi_\lambda) .$$

The isomorphism T can be write down explicitly.

By the above theorem, if we put $V_H(\lambda) = T(\underline{B}(H;\lambda))$, then we have a direct sum decomposition: $V(\chi_\lambda)=\oplus_{[H]\in Car(G)} V_H(\lambda)$. Let us define a representation of $W_H(\lambda)$ on $V_H(\lambda)$ in case where λ is regular.

<u>Proposition 2.3.</u> Let $\lambda \in \underline{h}^*_{\mathbb{C}}$ be <u>regular</u> and satisfy Assumption 2.1.

(1) For $w \in W_H(\lambda)$, we define an operator $\tau(w)$ on $\underline{B}(H;\lambda)$ as follows. For basis $\{\zeta(i,t;*)\}$, it operates as:

$$\tau(w)\zeta(i,t;*) = \zeta(i,tw^{-1};*) \quad (0 \leq i \leq k,\ t \in \widetilde{W}_H(\lambda)) .$$

Then τ defines a representation of $W_H(\lambda)$. Since $\underline{B}(H;\lambda)$ is isomorphic to $V_H(\lambda)$, we have a representation of $W_H(\lambda)$ on $V_H(\lambda)$.

(2) τ leaves $\underline{B}^i(H;\lambda)$ invariant and the representation $(\tau, \underline{B}^i(H;\lambda))$ is equivalent to a finite direct sum of representations induced from 1-dimensional representations.

(3) For an integral λ , we have $W_H(\lambda)=W$ and hence a representation of W on $V(\chi_\lambda)=\oplus V_H(\lambda)$ is defined. In this case, we have

$$V_H(\lambda) = \bigoplus_{0 \leq i \leq k} \mathrm{Ind}_{W(G;H_i)}^{W} \varepsilon(a_i;*) \quad .$$

When λ is integral, the representation $(\tau, V(\chi_\lambda))$ has another interpretation using tensor products with finite dimensional representations (Zuckerman[5], Barbasch-Vogan[1]). Our definition of the representation of $W_H(\lambda)$ is a generalization of it.

§3. Representations of Hecke rings. In the case λ is regular, we succeed in constructing the representation τ of integral Weyl groups. Then, how about singular λ ? In this section, we construct in three ways representations of Hecke rings on $V(\chi_\lambda)$, where λ is a singular infinitesimal character. We fix a Cartan subgroup H of G and singular $\lambda \in \underline{h}_{\mathbb{C}}^*$ which is dominant with respect to some ordering. At first, we introduce notions of Hecke rings. Let $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$ be the fixed subgroup of λ .

Lemma 3.1. W_λ is a subgroup of $W_H(\lambda)$.

Then a Hecke ring $\underline{H}(W_H(\lambda), W_\lambda)$ is defined as in [4]. In our case, since $W_H(\lambda)$ is a finite group, simpler interpretation is available. Namely $\underline{H}(W_H(\lambda), W_\lambda)$ is isomorphic as a $\mathbb{C}$-algebra to a subalgebra $e_\lambda \mathbb{C}[W_H(\lambda)] e_\lambda$ of the group ring $\mathbb{C}[W]$, where e_λ is an idempotent in $\mathbb{C}[W]$ defined by

$$e_\lambda = (\# W_\lambda)^{-1} \sum_{s \in W_\lambda} s \quad .$$

In the following, we always identify $\underline{H}(W_H(\lambda), W_\lambda)$ with $e_\lambda \mathbb{C}[W_H(\lambda)] e_\lambda$, a subalgebra of $\mathbb{C}[W]$. Now we explain three ways of constructing representations of $\underline{H}(W_H(\lambda), W_\lambda)$ on $V_H(\lambda)$.

Construction 1. Since $V_H(\lambda)$ is isomorphic to $\underline{B}(H;\lambda)$, we define a representation of $\underline{H}(W_H(\lambda), W_\lambda)$ on $\underline{B}(H;\lambda)$. Recall that $\zeta(i,t;*)$'s $(0 \leq i \leq k,\ t \in \widetilde{W}_H(\lambda))$ are generators for $\underline{B}(H;\lambda)$.

We define an operator $\sigma(e_\lambda w e_\lambda)$ $(w \in W_H(\lambda))$ on $\underline{B}(H;\lambda)$ as

$$\sigma(e_\lambda w e_\lambda)\, \zeta(i,t:*) = (\# W_\lambda)^{-1} \sum_{s \in W_\lambda} \zeta(i, tsw^{-1};*) \quad .$$

In this case $\zeta(i,t;*)$ is not necessarily a basis of $\underline{B}(H;\lambda)$, but $\sigma(e_\lambda w e_\lambda)$ extends to a well-defined linear operator on the space $\underline{B}(H;\lambda)$.

<u>Proposition 3.2.</u> $(\sigma, V_H(\lambda))$ defined as above is a representation of the Hecke ring $\underline{H}(W_H(\lambda), W_\lambda)$.

<u>Construction 2.</u> Choose an integral $\mu \in \underline{h}^*_{\mathbb{C}}$ such that $\lambda_0 = \lambda + \mu$ is dominant regular. Then we have

<u>Lemma 3.3.</u> (1) The subset $\widetilde{W}_H(\lambda_0)$ of W is equal to $\widetilde{W}_H(\lambda)$.
(2) The integral Weyl group $W_H(\lambda_0)$ is equal to $W_H(\lambda)$.

We have constructed a representation of $W_H(\lambda_0) = W_H(\lambda)$ on $V_H(\lambda_0)$ in §2. Put $U_H(\lambda_0) = \{v \in V_H(\lambda_0) \mid \tau(e_\lambda)v = 0\}$, where we regard τ as a representation of the group ring $\mathbb{C}[W_H(\lambda)]$. As a subalgebra of $\mathbb{C}[W_H(\lambda)]$, $\underline{H}(W_H(\lambda), W_\lambda)$ acts on $V_H(\lambda_0)/U_H(\lambda_0)$. We denote this representation of $\underline{H}(W_H(\lambda), W_\lambda)$ by $(\sigma, \mathrm{Red}^{W_H(\lambda)}_{W_\lambda} V_H(\lambda_0))$.

<u>Construction 3.</u> Let $\varphi^\lambda_{\lambda_0}: V(\lambda) \longrightarrow V(\lambda_0)$ and $\psi^{\lambda_0}_\lambda: V(\lambda_0) \longrightarrow V(\lambda)$ be Zuckerman's translation functors([9]). The functor φ is injective and ψ is surjective. These functors

preserve $V_H(\lambda)$ and $V_H(\lambda_0)$. For $e_\lambda w e_\lambda \in \underline{H}(W_H(\lambda), W_\lambda)$, we define $\sigma(e_\lambda w e_\lambda)$ by

$$\sigma(e_\lambda w e_\lambda)v=(\#W_\lambda)^{-1}\psi_\lambda^{\lambda_0}\circ\tau(e_\lambda w e_\lambda)\circ\varphi_{\lambda_0}^{\lambda}(v) \qquad (v\in V_H(\lambda)) .$$

Then $(\sigma, V_H(\lambda))$ is a representation of $\underline{H}(W_H(\lambda), W_\lambda)$.

Now we have the following theorem about the above three constructions.

<u>Theorem 3.4.</u> (1) The space $\mathrm{Red}_{W_\lambda}^{W_H(\lambda)} V_H(\lambda_0)$ is isomorphic to $V_H(\lambda)$ and $(\sigma, V_H(\lambda)) \simeq (\sigma, \mathrm{Red}\, V_H(\lambda_0))$ as a representation of $\underline{H}(W_H(\lambda), W_\lambda)$.

(2) The representations $(\sigma, V_H(\lambda))$'s of $\underline{H}(W_H(\lambda), W_\lambda)$ in Construction 1 and 3 are mutually equivalent.

§4. <u>Some applications.</u> In this section we state two applications of §§2 and 3. We use notations in the former sections.

4.1. <u>The number of irreducible representations.</u> If we put $U_H'(\lambda_0)=\{v\in V_H(\lambda_0) \mid \tau(e_\lambda)v=v\}$, then clearly it holds that $V_H(\lambda_0) = U_H'(\lambda_0)\oplus U_H(\lambda_0)$ (direct sum of vector spaces). Therefore we have $V_H(\lambda)\cong V_H(\lambda_0)/U_H(\lambda_0)\cong U_H'(\lambda_0)$. Put $n_H(\lambda)= \dim U_H'(\lambda_0)$.

<u>Corollary 4.1(to Theorem 3.4).</u> Using above notations, we have $\dim V(\chi_\lambda)= \sum_{[H]\in \mathrm{Car}(G)} n_H(\lambda)$.

Remark that $\dim V(\chi_\lambda)$ is the number of irreducible admissible representations of G which have infinitesimal character χ_λ. So, Corollary 4.1 says we are able to know the number of irreducible representations of G with singular infinitesimal character λ from the representations of integral Weyl groups at the regular infinitesimal character λ_0.

4.2. <u>τ-invariants.</u> From Theorem 3.4 (1), Red $V_H(\lambda_0)$ is isomorphic to $V_H(\lambda)$. Moreover, we know an isomorphism is given by $\psi_\lambda^{\lambda_0}: V_H(\lambda_0) \longrightarrow V_H(\lambda)$, i.e., $\ker \psi = U_H(\lambda_0)$. Now let us consider the case where $\# W_\lambda = 2$. In this case, since λ is dominant, $W_\lambda = \{e, s\}$ for some simple reflection s. Then we have the following corollary to Theorem 3.4.

<u>Corollary 4.2.</u> Let λ_0 be dominant regular element in $\underline{h}^*_{\mathbb{C}}$ such that $\langle \lambda_0, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$ for some simple root α. We put $\lambda = \lambda_0 - \frac{\langle \lambda_0, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$, a dominant singular element in $\underline{h}^*_{\mathbb{C}}$. Let Θ be a irreducible character with infinitesimal character λ_0. Then the two conditions below are mutually equivalent.

(1) $\psi_\lambda^{\lambda_0}(\Theta) = 0$.

(2) $\tau(s_\alpha)\Theta = -\Theta$, where s_α is the simple reflection with respect to α.

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