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UNITARY REPRESENTATIONS OF THE INFINITE SYMMETRIC GROUP

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INTRODUCTION

The *infinite symmetric group* $\mathfrak{S}_\infty$ is defined as the discrete group of all finite permutations of the set of all natural numbers. It was proved in Murray-von Neumann [5] that the group von Neumann algebra of $\mathfrak{S}_\infty$ is a factor of type II_1. Therefore it is a non-type I group, and probably for that reason, its irreducible (unitary) representations have not been investigated satisfactorily. In fact, only few results are known, for instance, see Lieberman [4] and Ol'shanskii [6]. On the other hand, factor representations of type II have been studied in Thoma [8], Vershik-Kerov [9],[10] and [11].

Our main purpose is to study irreducible representations of the infinite symmetric group as systematically as possible. The present note is divided into two parts: The first part (§1-§3) contains some results concerning irreducible representations discussed in [4] and [6]. In the second part (§4-§6), we construct a new family of irreducible representations and give some remarks.

§ 1. IRREDUCIBLE REPRESENTATIONS $\{\pi^\rho\}$ AND $\{\bar{\pi}^\rho\}$

Let X be the set of all natural numbers and $\mathfrak{S}_\infty$ the group of all finite permutations of X. If Y is a subset of X, we denote by $\mathfrak{S}(Y)$ the subgroup of all finite permutations of X which act identically outside Y. For brevity, we write $\mathfrak{S}_n$ and $\mathfrak{S}_{\infty-n}$ for $\mathfrak{S}(\{1,2,\dots,n\})$ and $\mathfrak{S}(\{n+1,n+2,\dots\})$, respectively.

If ρ is a (finite dimensional unitary) representation of $\mathfrak{S}_n$, we define unitary representations π^ρ and $\bar{\pi}^\rho$ of $\mathfrak{S}_\infty$ as the induced representations $\mathrm{Ind}_{\mathfrak{S}_n\times\mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty} \rho\otimes 1$ and $\mathrm{Ind}_{\mathfrak{S}_n\times\mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty} \rho\otimes\mathrm{sgn}$, respectively. Obviously, $\pi^\rho = 1$ (the trivial representation) and $\bar{\pi}^\rho = \mathrm{sgn}$ (the alternating representation) if ρ is the trivial representation of $\mathfrak{S}_0 = \{e\}$. The unitary representations π^ρ are discussed in Lieberman [4] and Ol'shanskii [6]. In view of their results, we obtain the following

PROPOSITION 1.1. (1) Let ρ be a representation of $\mathfrak{S}_n$. Then π^ρ is irreducible $\Longleftrightarrow$ $\bar{\pi}^\rho$ is irreducible $\Longleftrightarrow$ ρ is irreducible.

(2) Let ρ and ρ' be irreducible representations of $\mathfrak{S}_n$ and $\mathfrak{S}_{n'}$, respectively. Then

(2-1) $\pi^\rho \simeq \bar{\pi}^{\rho'} \Longleftrightarrow \pi^\rho \simeq \bar{\pi}^{\rho'} \Longleftrightarrow \rho \simeq \rho'$ (including $n = n'$),

(2-2) π^ρ is not equivalent to $\bar{\pi}^{\rho'}$.

For the structure of the representations $\{\pi^\rho\}$ we have the following result. An analoguous assertions for $\{\bar{\pi}^\rho\}$ are easily obtained and omitted.

PROPOSITION 1.2. Let ρ and ρ' be representations of $\mathfrak{S}_n$ and $\mathfrak{S}_{n'}$, respectively. Then we have

(1) $\pi^\rho \oplus \pi^{\rho'} \simeq \pi^{\rho\oplus\rho'}$ if $n = n'$,

(2) $\pi^\rho \otimes \pi^{\rho'} \simeq \sum_{j=0}^{n\wedge n'} \pi^{\tau(j)}$, where $\tau(j) = \mathrm{Ind}_{\mathfrak{S}_j\times\mathfrak{S}_{n-j}\times\mathfrak{S}_{n'-j}}^{\mathfrak{S}_{n+n'-j}} \rho''$ and ρ'' is defined as $\rho''(\sigma_1,\sigma_2,\sigma_3) = \rho(\sigma_1\sigma_2)\otimes\rho'(\sigma_1\sigma_3)$,

(3) Identifying $\mathfrak{S}_{\infty-n}$ with $\mathfrak{S}_\infty$ by shift, we have

$$\mathrm{Ind}_{\mathfrak{S}_n\times\mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty} \rho\otimes\pi^{\rho'} \simeq \sum_{\tau\in\widehat{\mathfrak{S}}_{n+n'}} [\rho\otimes\rho' : \mathrm{Res}_{\mathfrak{S}_n\times\mathfrak{S}_{n'}}^{\mathfrak{S}_{n+n'}} \tau\ ]\ \pi^\tau .$$

The assertions (1) and (2) are easy to see and (3) follows from the next lemma which is a consequence of transitivity of induced representations.

LEMMA 1.3. Let H be a subgroup of $\mathfrak{S}_{\infty-n}$ and τ an arbitrary representation of H. Then, for any representation ρ of $\mathfrak{S}_n$, we have

$$\mathrm{Ind}_{\mathfrak{S}_n\times\mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty} \rho\otimes(\mathrm{Ind}_H^{\mathfrak{S}_{\infty-n}}\tau) = \mathrm{Ind}_{\mathfrak{S}_n\times H}^{\mathfrak{S}_\infty} \rho\otimes\tau .$$

We shall now give a useful realization of π^ρ. We write $X^{[n]}$, $n\geqslant 0$, for the set of all ordered n-tuples $i = (i_1,i_2, \dots, i_n)$ consisting of distinct elements i_k of X. The groups $\mathfrak{S}_\infty$ and $\mathfrak{S}_n$ act on $X^{[n]}$ by means of the maps:

$$i \longmapsto gi = (g(i_1), g(i_2), \dots, g(i_n)), \qquad g\in\mathfrak{S}_\infty ,$$
$$i \longmapsto i\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}), \qquad \sigma\in\mathfrak{S}_n .$$

Let (ρ,W) be a representation of $\mathfrak{S}_n$ and H^ρ the Hilbert space of all W-valued functions f defined on $X^{[n]}$ satisfying $f(i\sigma) = \rho(\sigma)^{-1}f(i)$. Then π^ρ is realized as

$$(\pi^\rho(g)f)(i) = f(g^{-1}i) .$$

Here we add the following result.

PROPOSITION 1.4. Let $(T^n,\ell^2(X^{[n]}))$ be the unitary representation defined naturally by the action of $\mathfrak{S}_\infty$ on $X^{[n]}$. Then we have

$$T^n \simeq \sum_{\rho\in\widehat{\mathfrak{S}}_n} (\dim\rho)\ \pi^\rho .$$

Proof. Consider the representation $\pi^{\tilde\rho}$ with the regular representation $\tilde\rho$ of $\mathfrak{S}_n$. Q.E.D.

§ 2. REPRESENTATIONS ON PRODUCT SPACES

Let $\mathcal{Z}$ be a standard Borel space. The infinite symmetric group $\mathfrak{S}_\infty$ acts on $\mathcal{Z}^\infty$, the countably infinite product of $\mathcal{Z}$, by means of the maps:

$$z = (z_1, z_2, \ldots) \longmapsto gz = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, \ldots).$$

If μ is an $\mathfrak{S}_\infty$-quasi-invariant measure on $\mathcal{Z}^\infty$, the triple $(\mathfrak{S}_\infty, \mathcal{Z}^\infty, \mu)$ is called a *dynamical system* after Kirillov [3]. It would be very interesting to consider the unitary representation U arising from the dynamical system.

We shall give an irreducible decomposition of the unitary representation $(U, L^2(\mathcal{Z}^\infty, \mu))$ in case when μ is an $\mathfrak{S}_\infty$-invariant probability measure. By virtue of Hewitt and Savage [2], we have only to consider the case when μ is a product measure of an identical probability measure μ_1 on $\mathcal{Z}$.

We put

$$\mathbb{N}_0^\infty = \left\{ j = (j_1, j_2, \ldots) \;;\; \begin{array}{l} j_\ell \text{ a non-negative integer and} \\ j_\ell = 0 \text{ except finitely many } \ell \end{array} \right\}$$

and

$$\mathbb{N}_{k,0}^\infty = \{ j = (j_1, j_2, \ldots) \in \mathbb{N}_0^\infty \;;\; j_\ell \leqslant k \text{ for all } \ell \},\ k \geqslant 0.$$

The group $\mathfrak{S}_\infty$ acts on $\mathbb{N}_0^\infty$ and $\mathbb{N}_{k,0}^\infty$ in the same manner as above.

PROPOSITION 2.1. U is equivalent to the unitary representation corresponding to the $\mathfrak{S}_\infty$-space $\mathbb{N}_{k,0}^\infty$ or $\mathbb{N}_0^\infty$ according to dim $L^2(\mathcal{Z}, \mu_1) = k+1$ or ∞.

Proof. For a fixed orthogonal basis $\{ f_0 \equiv 1, f_1, f_2, \ldots \}$ for $L^2(\mathcal{Z}, \mu_1)$ we can construct an orthogonal basis $\{ f_j \;;\; j \in \mathbb{N}_{k,0}^\infty \text{ (or } \mathbb{N}_0^\infty)\}$ for $L^2(\mathcal{Z}^\infty, \mu_1^\infty)$ such that $U(g) f_j = f_{gj}$. Q.E.D.

Thus it is sufficient to consider the $\mathfrak{S}_\infty$-spaces $\mathbb{N}_{k,0}^\infty$ and $\mathbb{N}_0^\infty$. For $n \geqslant 1$ we denote by $\mathcal{P}(n)$ the set of all partitions of n:

$$\mathcal{P}(n) = \{ d = (d_1, \ldots, d_n) \;;\; d_j \text{ non-negative integer with } \textstyle\sum j d_j = n \}.$$

For $d \in \mathfrak{Y}(n)$, $Y(d)$ denotes a Young subgroup of $\mathfrak{S}_n$ of the form:

$$Y(d) = \underbrace{\mathfrak{S}(\{\cdot\}) \times \ldots \times \mathfrak{S}(\{\cdot\})}_{d_1} \times \underbrace{\mathfrak{S}(\{\cdot\cdot\}) \ldots \times \mathfrak{S}(\{\cdot\cdot\})}_{d_2} \quad \ldots$$

THEOREM 2.2. U is decomposed as follows:

$$U \simeq 1 \oplus \sum_{n \geqslant 1} \sum_{d \in \mathfrak{Y}(n)} m(d)\ \pi^{\rho(d)} ,$$

where $\rho(d) = \mathrm{Ind}_{Y(d)}^{\mathfrak{S}_n} 1$ and $m(d) = \binom{k}{d_1 + \ldots + d_n} \dfrac{(d_1 + \ldots + d_n)!}{d_1! \ldots d_n!}$, $d \in \mathfrak{Y}(n)$,

$k+1 = \dim L^2(\mathfrak{Z}, \mu_1)$, with the conventon that $m(d) = \infty$ (countably infinite) if $\dim L^2(\mathfrak{Z}, \mu_1) = \infty$.

Proof. First we decompose the $\mathfrak{S}_\infty$-spaces $\mathbb{N}_{k,0}^{\infty}$ and $\mathbb{N}_0^{\infty}$ into a disjoint union of $\mathfrak{S}_\infty$-orbits $\mathcal{O}$. Next we consider the representation corresponding to the transitive $\mathfrak{S}_\infty$-space $\mathcal{O}$. We skip the detailed discussions. Q.E.D.

Needless to say, $\pi^{\rho(d)}$ is decomposed into a finite sum of irreducible representations π^{ρ} by Proposition 1.2. Finally we remark that the dynamical system $(\mathfrak{S}_\infty, \mathfrak{Z}^\infty, \mu_1^\infty)$ is not recovered from the corresponding unitary representation. In fact we can prove the following

PROPOSITION 2.3. Let μ_1 and μ_1' be two probability measures on $\mathfrak{Z}$. Then two dynamical systems $(\mathfrak{S}_\infty, \mathfrak{Z}^\infty, \mu_1^\infty)$ and $(\mathfrak{S}_\infty, \mathfrak{Z}^\infty, \mu_1'^\infty)$ are isomorphic if and only if there exists a Borel isomorphism (defined up to null sets) $\psi : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ with $\psi\mu_1 = \mu_1'$.

§ 3. EXTENSION OF THE $\mathfrak{S}_\infty$-SPACE $X^{[n]}$ AND CORRESPONDING REPRESENTATIONS

Let Aut(X) denote the group of all permutations of X. Obviously, $\mathfrak{S}_\infty$ is a normal subgroup of Aut(X). First of all, we shall give a description of End($\mathfrak{S}_\infty$), the set of all endomorphisms of $\mathfrak{S}_\infty$.

THEOREM 3.1. For each $f \in \mathrm{End}(\mathfrak{S}_\infty)$ there exist unordered pairs $\mathcal{O}_{2,k} = \{j_{k1}, j_{k2}\}$ $(k=1,2,\ldots,s)$ and ordered sequences $\mathcal{O}_{\infty,\ell} = \{i_{\ell 1}, i_{\ell 2}, \ldots\}$ $(\ell=1,2,\ldots,t)$, possibly $s=0$ or $t=0$, satisfying the following properties:

(i) $\mathcal{O}_{2,1},\ldots,\mathcal{O}_{2,k},\mathcal{O}_{\infty,1},\ldots,\mathcal{O}_{\infty,t}$ are mutually disjoint as subsets of X,

(ii) for any $g \in \mathfrak{S}_\infty$, we have

$$f(g) = \left(\prod_{k=1}^{s} (j_{k1}\ j_{k2}) \right)^{(1-\mathrm{sgn}\ g)/2} \prod_{\ell=1}^{t} \begin{pmatrix} i_{\ell 1} & i_{\ell 2} & \cdots \\ i_{\ell g(1)} & i_{\ell g(2)} & \cdots \end{pmatrix}.$$

The above result follows from routine observation of generators of $\mathfrak{S}_\infty$, so we skip the proof here. The next assertion is then immediate.

COROLLARY 3.2. For any automorphism f of $\mathfrak{S}_\infty$, there exists a unique $\alpha \in \mathrm{Aut}(X)$ such that $f(g) = \alpha g \alpha^{-1}$, $g \in \mathfrak{S}_\infty$. In particular, $\mathrm{Aut}(\mathfrak{S}_\infty)$ is isomorphic to Aut(X).

We shall now consider the extension of the $\mathfrak{S}_\infty$-space $X^{[n]}$. Generally speaking, for an arbitrary set Z, the *first cohomology set* $H^1(\mathfrak{S}_\infty, X^{[n]}; \mathrm{Aut}(Z))$ describes extensions of the $\mathfrak{S}_\infty$-space $X^{[n]}$. For notations, see [3]. Here we shall restrict ourselves to those extensions which are described by $H^1(\mathfrak{S}_\infty, X^{[n]}; \mathfrak{S}_{\infty-n})$. Fix a cross section $i \mapsto g[i] \in \mathfrak{S}_\infty$ for the canonical projection $\mathfrak{S}_\infty \to X^{[n]} \simeq \mathfrak{S}_\infty / \mathfrak{S}_{\infty-n}$. For any $f \in \mathrm{End}(\mathfrak{S}_{\infty-n})$ we define a 1-cocycle $\beta_f \in Z^1(\mathfrak{S}_\infty, X^{[n]}; \mathfrak{S}_{\infty-n})$ by $\beta_f(g,i) = f(g[i]g\ g[g^{-1}i])$. Then we have the following

PROPOSITION 3.3. The map $f \mapsto \beta_f$ induces a surjection from $\mathrm{End}(\mathfrak{S}_{\infty-n})$ onto $H^1(\mathfrak{S}_\infty, X^{[n]}; \mathfrak{S}_{\infty-n})$. Moreover β_f and $\beta_{f'}$ are cohomologuous if and only if f and f' are conjugate in $\mathrm{Aut}(\{n+1, n+2, \ldots\})$.

Finally we shall consider the corresponding unitary representations. Let n and m be two non-negative integers. We put $Z = \{n+1, n+2, \ldots\} \subset X$. For any $\beta_f \in Z^1(\mathfrak{S}_\infty, X^{[n]}; \mathfrak{S}_{\infty-n})$, $f \in \mathrm{End}(\mathfrak{S}_{\infty-n})$, we define an action of $\mathfrak{S}_\infty$ on $X^{[n]} \times Z^{[m]}$ by means of the maps:

$$(i,j) \longmapsto g(i,j) = (gi, \quad \beta_f(g^{-1},i)^{-1}j), \quad g \in \mathfrak{S}_\infty.$$

We denote by $U^{n,m,f}$ the corresponding unitary representation. Viewing Proposition 3.3, if f and f' are conjugate in Aut(Z), $U^{n,m,f}$ and $U^{n,m,f'}$ are equivalent.

THEOREM 3.4. If f is an automorphism of $\mathfrak{S}_\infty$, we have

$$U^{n,m,f} \simeq \sum_{\rho \in \widehat{\mathfrak{S}_{n+m}}} (\dim \rho)\ \pi^\rho \simeq T^{n+m}$$

with the notation in Proposition 1.4.

§ 4. GENERAL RESULTS FOR DISCRETE GROUPS

Let G be a discrete group, H a subgroup and $\Omega = G/H$ the quotient space. We denote by $\omega_0 \in \Omega$ the point whose isotropy group is H. For each unitary character χ of H we consider the induced representation $U^\chi = \mathrm{Ind}_H^G \chi$. It is convenient to adopt the following realization of U^χ.

Let $L^2(\Omega)$ be the Hilbert space of all square summable functions on Ω. We fix a cross section $\omega \longmapsto g[\omega] \in G$ for the canonical projection $g \longmapsto g\omega_0 \in \Omega$, $g \in G$. Then the induced representation U^χ is give by the formula:

$$(U^\chi(g)f)(\omega) = \chi(g[\omega]^{-1} g\, g[g^{-1}\omega])\ f(g^{-1}\omega), \qquad f \in L^2(\Omega) \text{ and } g \in G.$$

PROPOSITION 4.1. Assume that all H-orbits in Ω are infinite sets except $\{\omega_0\}$. Then we have

(1) U^χ is irreducible,

(2) U^χ is equivalent to $U^{\chi'}$ if and only if $\chi = \chi'$.

Proof. Take an orthonormal basis $\{\delta_\omega ; \omega \in \Omega\}$, where δ_ω denotes the delta-function concentrated at ω. Suppose that T is a bounded operator on $L^2(\Omega)$ satisfying $U^\chi(g)T = TU^\chi(g)$ for all $g \in G$. Then one can see that $|T\delta_{\omega_0}(\omega)|$ is constant on each H-orbit in Ω. By assumption we conclude that T is a scalar operator. This proves (1). The assertion (2) is shown in a similar manner. Q.E.D.

By mimicking the above proof we have the following

PROPOSITION 4.2. Let α be an automorphism of G. If $|H : \alpha(gHg^{-1}) \cap H| = +\infty$ for all $g \in G$, two unitary representations U^χ and $U^{\chi'} \circ \alpha$ are disjoint for any unitary characters χ and χ' of H.

Several authors gave analoguous results for their own purposes. We refer to Godement [1], Saito [7] and Yoshizawa [12].

§ 5. IRREDUCIBLE REPRESENTATIONS $\{U^{\theta,\chi}\}$

Using the results of the previous section, we give a new family of irreducible representations of $\mathfrak{S}_\infty$. For any $\theta \in \mathrm{Aut}(X)$ we denote by $H(\theta)$ the subgroup of all finite permutations which commute with θ:

$$H(\theta) = \{ g \in \mathfrak{S}_\infty ; g\theta = \theta g \} \quad (= H(\theta^{-1})).$$

In what follows we shall restrict ourselves to some special automorphisms. For any integer $p \geq 2$, we denote by $\mathrm{Aut}_p(X)$ the set of all automorphisms $\theta \in \mathrm{Aut}(X)$ having the following two properties:

(i) $\theta = \prod_m (i_{m0}\ i_{m1}\ \cdots\ i_{m\,p-1})$ in cycle-notation,

(ii) $\mathrm{supp}\,\theta = X$, i.e. no point of X is fixed by θ.

Let $A(\theta)$ be the abelian subgroup of $\mathfrak{S}_\infty$ which is generated by all the cyclic permutations $(i_{m0}\ i_{m1}\ \cdots\ i_{m\,p-1})$, m=1,2,.... And let $S(\theta)$ be the subgroup of all permutations $g \in \mathfrak{S}_\infty$ satisfying the property: there exists some $\sigma \in \mathfrak{S}_\infty$ such that $g(i_{mk}) = i_{\sigma(m)k}$ for all m=1,2,... and k=0,1,. ...,p-1. As is easily seen, $A(\theta)S(\theta) = S(\theta)A(\theta) = S(\theta) \ltimes A(\theta)$ (semidirect product).

PROPOSITION 5.1. $H(\theta) = S(\theta) \ltimes A(\theta)$.

The proof is skipped for want of space. The next result describes the structure of $H(\theta)$ for an arbitrary automorphism $\theta \in \mathrm{Aut}(X)$.

PROPOSITION 5.2. Any $\theta \in \mathrm{Aut}(X)$ admits an expression of the form: $\theta = \theta_\infty \theta_2 \theta_3$ ....., where θ_n is a product of disjoint cycles of length n and supp θ_n ($n=\infty,2,3,\ldots$) mutually disjoint. Furthermore we have $H(\theta) = \mathfrak{S}(X\text{-supp }\theta) \times H'(\theta_2) \times H'(\theta_3) \times$ ....., in the sence of restricted direct product. Here $H'(\theta_k)$ denotes the subgroup of all permutations in $\mathfrak{S}(\text{supp }\theta_k)$ which commute with θ_k.

Let $\theta \in \mathrm{Aut}_p(X)$ with $p \geqslant 2$. It follows from Proposition 5.1 that $H(\theta)$ admits exactly 2p unitary characters. (Note that $S(\theta) \simeq \mathfrak{S}_\infty$ has exactly two unitary characters 1 and sgn.) By Proposition 4.1 we have the following

THEOREM 5.3. Let $\theta \in \mathrm{Aut}_p(X)$ with $p \geqslant 2$.

(1) For any unitary character χ of $H(\theta)$, $U^{\theta,\chi}$ is irreducible.

(2) For two unitary characters χ and χ' of $H(\theta)$, $U^{\theta,\chi}$ is equivalent to $U^{\theta,\chi'}$ if and only if $\chi = \chi'$.

For two automorphisms $\theta = \Pi\,(i_{m0} \ldots i_{m\,p-1}) \in \mathrm{Aut}_p(X)$ and $\theta' = \Pi\,(j_{n0} \ldots j_{n\,p'-1}) \in \mathrm{Aut}_{p'}(X)$ we denote by $N(\theta,\theta')$ the number of pairs (m,n) such that $\{i_{m0},\ldots,i_{m\,p-1}\} = \{j_{n0},\ldots,j_{n\,p'-1}\}$.

THEOREM 5.4. If $N(\theta,\theta')$ is finite, two unitary representations $U^{\theta,\chi}$ and $U^{\theta',\chi'}$ are not equivalent for any unitary characters χ and χ' of $H(\theta)$ and $H(\theta')$, respectively.

Proof (in case of $p=p'$). First we note that $|H(\theta) : H(\theta)\cap H(\theta')| = \infty$ by assumption. We choose $\alpha \in \mathrm{Aut}(X)$ such that $\theta' = \alpha\theta\alpha^{-1}$ and denote by $\hat{\alpha}$ the automorphism of $\mathfrak{S}_\infty$ defined by $\hat{\alpha}(g) = \alpha g\alpha^{-1}$, $g \in \mathfrak{S}_\infty$. Then one can show that $|H(\theta) : \hat{\alpha}(gH(\theta)g^{-1}) \cap H(\theta)| = \infty$. Applying Proposition 4.2, we get the desired result. The proof for $p\neq p'$ is similar. Q.E.D.

THEOREM 5.5. Let $\theta \in \mathrm{Aut}_p(X)$ with $p \geq 2$ and χ a unitary character of $H(\theta)$. Then $U^{\theta,\chi}$ is equivalent to neither π^ρ nor $\bar{\pi}^\rho$ for any irreducible representation ρ of $\mathfrak{S}_n$, $n=0,1,2,\ldots$.

Proof. We have only to repeat the arguments in Propositions 4.1 and 4.2. Q.E.D.

§ 6. CONCLUDING REMARKS

In this note we restricted ourselves to rather special kind of automorphisms, namely, $\mathrm{Aut}_p(X)$, and discussed the unitary representations $U^{\theta,\chi}$. However, with the help of Proposition 5.2, we might discuss them in more general situation. Two remarks are now in order.

If θ has a finite support, i.e. $\theta \in \mathfrak{S}_\infty$, $H(\theta)$ admits a direct product decomposition $H(\theta) = H'(\theta) \times \mathfrak{S}(X\text{-supp}\,\theta)$, where $H'(\theta) \subset \mathfrak{S}(\mathrm{supp}\,\theta)$. Then the next result is easy to see.

$$\mathrm{Ind}_{H(\theta)}^{\mathfrak{S}_\infty} \rho'\otimes 1 = \sum_{\rho\in \hat{\mathfrak{S}}(\mathrm{supp}\,\theta)} [\rho' : \mathrm{Res}_{H'(\theta)}^{\mathfrak{S}(\mathrm{supp}\,\theta)} \rho]\ \pi^\rho,$$

where ρ' is a representation of $H'(\theta)$. Hence our method yields the irreducible representations which are discussed in [4] and [6].

Next assume that $\theta \in \mathrm{Aut}(X)$ admits a cycle-notation $\Pi\,(i_{m0} \cdots i_{m\,p-1})$ with X-supp $\theta = \{1,2,\ldots,n\}$. Since $H(\theta)$ is a direct product $\mathfrak{S}_n \times H'(\theta)$, any unitary character is of the form $\varepsilon \otimes \chi$, where $\varepsilon = 1$ or sgn. Then we see that

$$\mathrm{Ind}_{H(\theta)}^{\mathfrak{S}_\infty}\ \varepsilon \otimes \chi = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_{\infty-n}}^{\mathfrak{S}_\infty}\ \varepsilon \otimes U^{\theta,\chi},$$

where $\mathfrak{S}_{\infty-n}$ is identified with $\mathfrak{S}_\infty$. Thus the problem is reduced to the study of the representations stated in Lemma 3.1.

Before closing this note, I would like to pose four problems which seem important and interesting.

(1) Study of the unitary representations arising from the dynamical system $(\mathfrak{S}_\infty, \mathfrak{Z}^\infty, \mu)$ with a quasi-invariant measure μ.

(2) Complete classification of the unitary representations $U^{\theta,\chi}$.

(3) Irreducible decompositions of factor representations of type II.

(4) Systematic study of irreducible representations of discrete groups.

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