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Closure relation for orbits on affine symmetric spaces under the action of minimal parabolic subgroups

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Let G be a connected Lie group, σ an involutive automorphism of G and H a subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$ where $G^\sigma = \{x \in G \mid \sigma x = x\}$ and G_0^σ is the connected component of G^σ containing the identity. Then the factor space $H\backslash G$ is called an affine symmetric space. We assume that G is real semisimple throughout this paper.

Let P^0 be a minimal parabolic subgroup of G. Then a parametrization of the double coset decomposition $H\backslash G/P^0$ is given in [1] and [2]. In this note we study the closure relations for the double coset decomposition.

Let $\underline{g}$ be the Lie algebra of G and σ the automorphism of $\underline{g}$ induced from the automorphism σ of G. Let θ be a Cartan involution of $\underline{g}$ such that $\sigma\theta = \theta\sigma$. Let $\underline{g} = \underline{h} + \underline{q}$ (resp. $\underline{g} = \underline{k} + \underline{p}$) be the decomposition of $\underline{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ).

Let x be an arbitrary element of G. By Theorem 1 in [1], there exists an $h \in G_0^\sigma$ such that $P = hxP^0x^{-1}h^{-1}$ can be written as

$$P = P(\underline{a}, \Sigma^+) = Z_G(\underline{a}) \exp \underline{n}$$

where $\underline{a}$ is a σ-stable maximal abelian subspace of $\underline{p}$, Σ^+ is a positive system of the root system Σ of the pair $(\underline{g}, \underline{a})$, $Z_G(\underline{a})$ the centralizer of $\underline{a}$ in G and $\underline{n} = \sum_{\alpha\in\Sigma^+} \underline{g}(\underline{a}; \alpha)$. ($\underline{g}(\underline{a}; \alpha) = \{X \in \underline{g} \mid [Y, X] = \alpha(Y)X$ for all $Y \in \underline{a}\}$.) Since $(HxP^0)^{cl} = (HP)^{cl}hx$, we have only to study $(HP)^{cl}$.

Let K be the analytic subgroup of G for $\underline{k}$ and put $H^a = (K \cap H)\exp(\underline{p} \cap \underline{q})$. Then $H^a\backslash G$ is called the affine symmetric space associated to $H\backslash G$ (see [1]). Let $(H^aP)^{op}$ denote the minimal H^a-P invariant open subset containing H^aP. Clearly $(H^aP)^{op} = \bigcup_{H^aP\subset(H^ayP)^{cl}} H^ayP$. For every root α in Σ^+, put $\underline{a}^\alpha = \{Y \in \underline{a} \mid \alpha(Y) = 0\}$, $L_\alpha = Z_G(\underline{a}^\alpha)$ and choose an element w_α of $L_\alpha \cap N_K(\underline{a})$ which is not contained in $Z_K(\underline{a})$. Then $\mathrm{Ad}(w_\alpha)|_{\underline{a}}$ is the reflection with respect to α.

Theorem 1. Let $P = P(\underline{a}, \Sigma^+)$ be a parabolic subgroup of G such that $\sigma\underline{a} = \underline{a}$. Put $C = \{\alpha \in \Sigma^+ \mid \sigma\alpha \notin \Sigma^+,\ \sigma\alpha \neq -\alpha$ and $\frac{1}{2}\alpha \notin \Sigma\}$ and $n = \frac{1}{2}$(number of roots in C). Then we have the followings.

(i) There exists a sequence $\beta_1, \dots, \beta_n$ of roots in C satisfying the following two conditions.

(a) β_i is a simple root of the positive system $\Sigma^+_{i-1} = w_{\beta_{i-1}}\cdots w_{\beta_1}\Sigma^+$ and $\sigma\beta_i \notin \Sigma^+_{i-1}$ for $i = 1, \dots, n$.

(b) $C = \{\beta_1, \dots, \beta_n, -\sigma\beta_1, \dots, -\sigma\beta_n\}$.

In the followings we fix a sequence $\beta_1, \dots, \beta_n$ of roots in (i). Define simple roots α_i of Σ^+ by $\alpha_i = w_{\beta_1}\cdots w_{\beta_{i-1}}\beta_i$ for $i = 1, \dots, n$ and put $w = w_{\beta_n}\cdots w_{\beta_1}$.

(ii) Let L be the analytic subgroup of G for $[\underline{z}_{\underline{g}}(\underline{a}\cap\underline{h}),$

$\underline{z}_{\underline{g}}(\underline{a}\cap\underline{h})]$. ($\underline{z}_{\underline{g}}(\underline{a}\cap\underline{h})$ is the centralizer of $\underline{a}\cap\underline{h}$ in $\underline{g}$.) Then

$$(HP)^{cl} = H((L\cap H)(L\cap P))^{cl} wL_{\alpha_n}\cdots L_{\alpha_1}P$$

and

$$(H^aP)^{op} = H^a((L\cap H^a)(L\cap P))^{op} wL_{\alpha_n}\cdots L_{\alpha_1}P.$$

Here $((L\cap H^a)(L\cap P))^{op}$ is the minimal $L\cap H^a$ - $L\cap P$ invariant open subset of L containing $(L\cap H^a)(L\cap P)$.

(iii) $(L\cap H)(L\cap P)$ is open in L and $(L\cap H^a)(L\cap P)$ is closed in L.

Remark 1. (i) Since L is a connected semisimple Lie subgroup of G such that $\sigma L = \theta L = L$, we can apply the theorems in [1] to the double coset decompositions $L\cap H\backslash L/L\cap P$ and $L\cap H^a\backslash L/L\cap P$.

(ii) If the number of the open double cosets in $L\cap H\backslash L/L\cap P$ is one (then the number of the closed double cosets in $L\cap H^a\backslash L/L\cap P$ is one), for instance when G is a complex semisimple Lie group and σ is a complex linear involution, then it is clear that

$$((L\cap H)(L\cap P))^{cl} = ((L\cap H^a)(L\cap P))^{op} = L.$$

Theorem 2. Retain the notations in Theorem 1. Let y be an element contained in $(HP)^{cl}$ (resp. $(H^aP)^{op}$). Then there exist $z_1 \in L_{\alpha_1}, \ldots, z_n \in L_{\alpha_n}$ and $z \in Lw$ satisfying the following four conditions.

(i) $z \in ((L\cap H)(L\cap P))^{cl}w$ (resp. $z \in ((L\cap H^a)(L\cap P))^{op}w$).

(ii) $HyP = Hzz_n \ldots z_1 P$ (resp. $H^a yP = H^a zz_n \ldots z_1 P$).

(iii) $\dim Hzz_n \ldots z_{i+1} P \leq \dim Hzz_n \ldots z_i P$ (resp. $\dim H^a zz_n \ldots z_1 P \geq \dim H^a zz_n \ldots z_1 P$) for $i = 1, \ldots, n$.

(iv) For every $i = 1, \ldots, n$, z_i is given as follows. Put $\underline{a}_i = \mathrm{Ad}(zz_n \ldots z_{i+1})\underline{a}$ and define a root $\alpha_i' \in \Sigma(\underline{a}_i)$ by $\alpha_i' = \alpha_i \circ \mathrm{Ad}(zz_n \ldots z_{i+1})^{-1}$. If $g(\underline{a}_i; \alpha_i') \cap \underline{q} = \{0\}$, then $z_i = 1$ or w_{α_i}. If $g(\underline{a}_i; \alpha_i') \cap \underline{q} \neq \{0\}$, then $z_i = 1$, w_{α_i}, c_{α_i} or $c_{\alpha_i}^{-1}$. Here c_{α_i} is an element of L_{α_i} defined by $c_{\alpha_i} = (zz_n \ldots z_{i+1})^{-1} c_{\alpha_i}' zz_n \ldots z_{i+1}$, $c_{\alpha_i}' = \exp \frac{\pi}{2}(X + \theta X)$ with $X \in \underline{g}(\underline{a}_i; \alpha_i') \cap \underline{q}$ satisfying $2\langle \alpha_i', \alpha_i' \rangle B(X, \theta X) = -1$. ($B(\ ,\ )$ is the Killing form on $\underline{g}$ and $\langle\ ,\ \rangle$ is the inner product on the dual of $\underline{a}$ induced from $B(\ ,\ )$.)

Example. When $G = G_1 \times G_1$, $H = \{(x, x) \mid x \in G_1\}$ and $P = P_1 \times P_1$ with a connected semisimple Lie group G_1 and a minimal parabolic subgroup $P_1 = P(\underline{a}_1, \Sigma_1^+)$ of G_1, the double coset decomposition $H\backslash G/P$ can be naturally identified with the decomposition $P_1\backslash G_1/P_1$. In this example we have $z_i = 1$ or w_{α_i} for $i = 1, \ldots, n$ and we can see easily that Theorem 2 implies the wellknown equivalence between the Bruhat order on the Weyl group $W(\underline{a}_1)$ and the closure relation for the Bruhat decomposition $G_1 = \bigcup_{w \in W(\underline{a}_1)} P_1 w P_1$.

Theorem 3. Retain the notations in Theorem 1.

(i) Let HyP be a closed double coset. Then HyP is contained in $(HP)^{cl}$ if and only if $HyP \subset HzW_{\alpha_n} \ldots W_{\alpha_1} P$ with some $z \in Lw$ such that $(L \cap H)zw^{-1}(L \cap P)$ is closed in L. Here W_{α_i}

$\{1, w_{\alpha_i}\}$ for $i = 1,\dots, n$.

(ii) Let $H^a yP$ be an open double coset. Then $H^a yP$ is contained in $(H^a P)^{op}$ if and only if $H^a yP \subset H^a z W_{\alpha_n} \dots W_{\alpha_1} P$ with some $z \in Lw$ such that $(L \cap H^a) z w^{-1} (L \cap P)$ is open in L.

Remark 2. By Theorem 1 and Theorem 2, the study of $(HP)^{cl}$ is reduced to that of $((L \cap H)(L \cap P))^{cl}$. On the other hand, we can find by Theorem 3 (ii) all the $L \cap H$ - $L \cap P$ double cosets contained in $((L \cap H)(L \cap P))^{cl}$ in the following way. Let $(L \cap H) y (L \cap P)$ be an arbitrary $L \cap H$ - $L \cap P$ double coset in L. We may assume that $Ad(y)\underline{a}$ is σ-stable by [1] Theorem 1. Then considering L, $L \cap H$ and $y(L \cap P)y^{-1}$ as G, H^a and P in Theorem 3 (ii), respectively, we can see whether $(L \cap H)(L \cap P)$ is contained in $((L \cap H)y(L \cap P))^{op}$ or not (whether $(L \cap H)y(L \cap P)$ is contained in $((L \cap H)(L \cap P))^{cl}$ or not).

We can also find all the H^a-P double cosets contained in $(H^a P)^{op}$ by Theorem1, Theorem 2 and Theorem 3 (i) in the similar way.

From this remark we have the following.

Corollary. Let HxP be an arbitrary H-P double coset and choose a $y \in HxP$ so that $Ad(y)\underline{a}$ is σ-stable. Then $HyP \subset (HP)^{cl}$ if and only if $H^a P \subset (H^a yP)^{cl}$.

Theorem 1 and Theorem 2 are reduced to lemmas which are generalizations of [3] Lemma 5.1. On the other hand Theorem 3 is reduced to the following.

Proposition. For any closed double coset HxP and any open double coset HyP, we have $HxP \subset (HyP)^{cl}$.

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