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Remarks on involutions on a root system

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はじめに，1983.11.16－11.19 の表現論シンポジウムでの著者の講演内容とこの報告とは無関係であることをお断りしておきます。

この報告の内容は，root 系の involution について最近調べたことをまとめたものです。[OS] を参照されれば，ここで導入した R-root system の意義がわかるでしょう。

root系の一般の involution については杉浦先生が調べられています（[S]）。一般の involution と実半単純 Lie 環の Cartan subalgebras の共役類とが関係しており，それについても言及する予定でした。しかし，紙面の都合，結果に少し不完全な所もあり，またオランダの A.G. Helminck が国際数学者会議で類似の結果を報告している，などの理由で，これに掲載するのは差し控えました。

§1. The Satake diagram for an involution of a root system.

Let Σ be a root system on a real vector space V. Unless otherwise stated, Σ is assumed to be reduced, that is, $\frac{1}{2}\alpha \notin \Sigma$ for any $\alpha \in \Sigma$. Let $A(\Sigma)$ be the automorphism group of Σ. Let $\langle\ ,\ \rangle$ be an $A(\Sigma)$-invariant inner product on Σ. As usual, $W = W(\Sigma)$ denotes the Weyl group of Σ. Clearly $W \subset A(W)$. If Ψ is a fundamental system for Σ, then $\Sigma^+(\Psi)$ denotes the set of positive roots for the order defined by Ψ.

Definition 1.1. If σ is an involution of Σ, then the pair (Σ, σ) is called a root system with an involution.

Let (Σ, σ) be a root system with an involution. Put $V^{\pm} = \{v \in V;\ \sigma v = \pm v\}$ and $\Sigma_0 = V^+ \cap \Sigma$.

Definition 1.2. Let (Σ, σ) be a root system with an involution. Let Ψ be a fundamental system for Σ. Then Ψ is called a σ-fundamental system if σ leaves $\Sigma^+(\Psi)-\Sigma_0$ invariant.

Let (Σ, σ) be a root system with an involution and let Ψ be a σ-fundamental system for Σ. Then for the triple (Σ, Ψ, σ), a diagram $S(\Sigma, \Psi, \sigma)$ is defined by the same way as the Satake diagram is done for a normal $(-\sigma)$-system of roots.

Definition 1.3. The diagram $S(\Sigma, \Psi, \sigma)$ is called the

Satake diagram for the root system with an involution (Σ, σ).

§2. N-root systems.

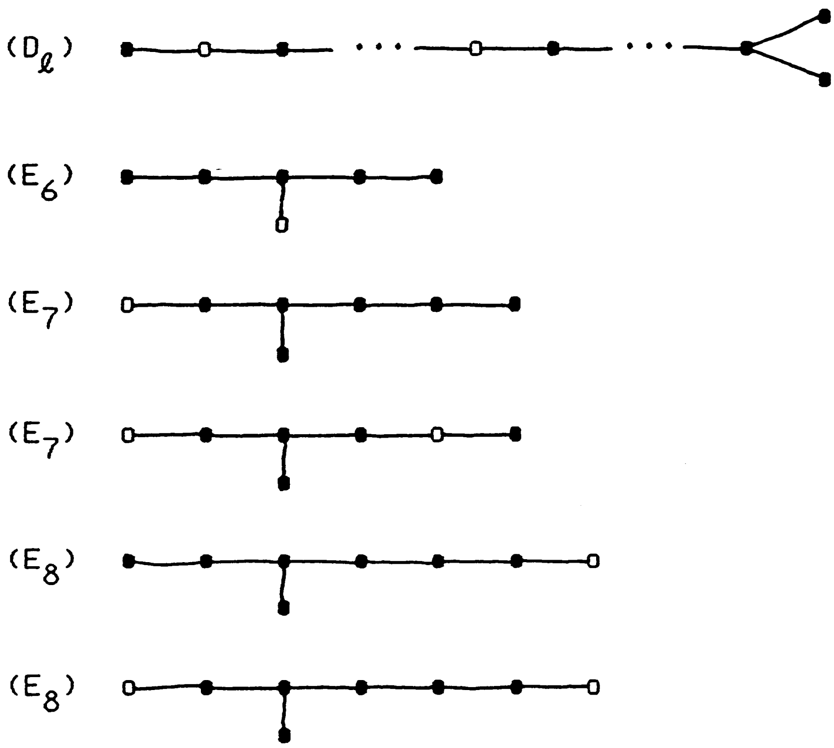
Definition 2.1. Let (Σ, σ) be a root system with an involution. Then (Σ, σ) is called an N-root system (with an involution) if $\sigma\alpha + \alpha \in\!\!\!\!/\ \Sigma$ for any $\alpha \in \Sigma$.

Remark 2.2. By definition, a root system with an involution (Σ, σ) is an N-root system if and only if Σ is $(-\sigma)$-normal in the sense of [Ar].

Let $\underline{g}$ be a real semisimple Lie algebra and let $\underline{g} = \underline{k} + \underline{p}$ be a Cartan decomposition of $\underline{g}$. Let θ be the corresponding Cartan involution. Let $\underline{a}$ be a maximal abelian subspace of $\underline{p}$ and let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ containing $\underline{a}$. Then, by definition, $\theta\underline{h} = \underline{h}$. Let Σ be the root system of $(\underline{g}, \underline{h})$. Then Σ is also regarded as the root system on the real vector space $V = \sqrt{-1}(\underline{k} \cap \underline{h}) + \underline{p} \cap \underline{h}$. Since θ leaves V invariant, we may put $\sigma = \theta | V$. Then it follows from [Ar] that (Σ, σ) is an N-root system with an involution.

Definition 2.3. An N-root system (Σ, σ) is called an NE-root system (with an involution) if (Σ, σ) is obtained from a semisimple Lie algebra as in the above way.

The classification of NE-root systems is equivalent to that of real forms of complex semisimple Lie algebras. From this stand point, Araki [Ar] accomplished the classification of NE-root systems. But, as was noted in [Ar], every N-root system is not an NE-root system. The classification of N-root systems is essentially given in [Ar], but this is not explicitly written there. By this reason, we now give the list of such N-root systems (Σ, σ) that Σ is σ-irreducible and that (Σ, σ) is not an NE-root system. The result is given in the following table.



In the rest of this section, we discuss the condition introduced in [OS, §3].

Let Σ be a root system and let σ, σ' be involutions of Σ. Then consider the following conditions:

(A0) (Σ, σ) and (Σ, σ') are N-root systems.

(A1) $\sigma\sigma' = \sigma'\sigma$.

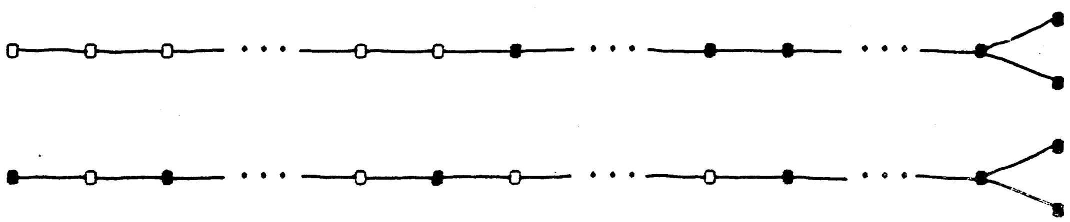
(A2) If $\alpha \in \Sigma$ satisfies the condition $(1-\sigma)(1-\sigma')\alpha = 0$, then $\sigma\alpha = \alpha$ or $\sigma'\alpha = \alpha$ holds.

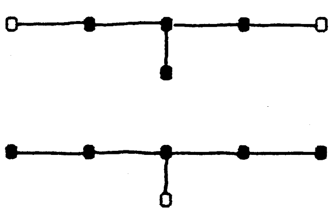
Discuss the relation between these conditions and the Satake diagrams.

<u>Propostition 2.4</u>. Let (Σ, σ) and (Σ, σ') be N-root systems. Assume that there exists a fundamental system Ψ for Σ such that Σ is both σ-fundamental and σ'-fundamental. Then the conditions (A1) and (A2) are equivalent.

The proof is based on the classification of N-root systems and is accomplished by case by case examination.

<u>Remark 2.5</u>. Let (Σ, σ), (Σ, σ') be one of the followings.

I) 

II) 

Then $\sigma\sigma' \neq \sigma'\sigma$. The remaining cases are satisfied with the conditions in Proposition 2.4.

Let Σ be a root system and let σ, σ' be involutions of Σ. Assume that (Σ, σ) and (Σ, σ') are N-root systems and that (A1) and (A2) hold. Put $\Sigma(\sigma) = \{\frac{1}{2}(\alpha-\sigma\alpha): \alpha \in \Sigma\} - \{0\}$. Then it follows from [Ar] that $\Sigma(\sigma)$ is a root system. Moreover, since $\sigma\sigma' = \sigma'\sigma$, the involution σ' leaves $\Sigma(\sigma)$ invariant. Hence $(\Sigma(\sigma), \sigma')$ is a root system with an involution. In genral, $(\Sigma(\sigma), \sigma')$ is not an N-root system. But $(\Sigma(\sigma), \sigma')$ is satisfied with the following condition:

(*) If $\lambda \in \Sigma(\sigma)$ satisfies that $\langle\lambda, \sigma'\lambda\rangle < 0$, then $\sigma'\lambda = -\lambda$.

(Cf. [OS, §3]). Noting this, we define the following.

Definition 2.6. Let (Σ, σ) be a root system with an involution. Assume that Σ is not necessarily reduced. Then (Σ, σ) is an R-root system (with an involution) if the condition (**) holds:

(**) If $\alpha \in \Sigma$ satisfies that $\langle\alpha, \sigma\alpha\rangle < 0$, then $\sigma\alpha = -\alpha$.

Remark 2.7. (1) If (Σ, σ) is an N-root system, then (Σ, σ) is also an R-root system.

(2) If (Σ, σ) is an R-root system, then $\Sigma(\sigma)$ is also a root system.

(3) Let (Σ, σ) and (Σ, σ') be N-root systems. Then both $(\Sigma(\sigma), \sigma')$ and $(\Sigma(\sigma'), \sigma)$ are R-root systems. Moreover, if $\Sigma(\sigma,\sigma') = \{\frac{1}{4}(1-\sigma)(1-\sigma')\alpha;\ \alpha \in \Sigma\} - \{0\}$, then $\Sigma(\sigma,\sigma')$ is a root system.

§3. The classification of R-root systems with involutions.

In this section, we classify R-root systems with involutions.

First give a lemma.

Lemma 3.1. Let (Σ,σ) be an R-root system. Assume that Σ is simply-laced or is doubly-laced of type (1:3) (for the definition of these terminologies, see [Ar]). Then (Σ, σ) is an N-root system.

Proof. We assume that there exists an element α of Σ such that $\alpha + \sigma\alpha \in \Sigma$ and lead a contradiction.

First consider the case where Σ is simply-laced. Then since α and $\alpha+\sigma\alpha$ are of the same length, we have

$$\langle\alpha, \alpha\rangle = \langle\alpha+\sigma\alpha, \alpha+\sigma\alpha\rangle = 2\langle\alpha, \alpha\rangle + 2\langle\alpha, \sigma\alpha\rangle.$$

This implies that $2\langle\alpha, \sigma\alpha\rangle = -\langle\alpha, \alpha\rangle < 0$. Since (Σ, σ) is an

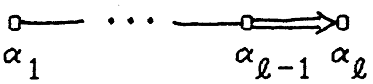
R-root system, we find that $\sigma\alpha = -\alpha$ and $0 = \alpha + \sigma\alpha \in \Sigma$. This contradicts the assumption on α.

Next consider the case where Σ is doubly-laced of type (1:3). In this case $\langle\alpha+\sigma\alpha, \alpha+\sigma\alpha\rangle = k\langle\alpha, \alpha\rangle$, where k is one of 1, 3 and $\frac{1}{3}$. Then it follows that $2\frac{\langle\alpha, \sigma\alpha\rangle}{\langle\alpha, \alpha\rangle} = k-2$. If $k = 1$, by the same argument as above, we lead a contradiction. Hence due to [W, p.10], we find that k must be 3. Then $\langle\alpha, \alpha\rangle = 2\langle\alpha, \sigma\alpha\rangle$. Let s_α be the reflection with respect to α. Then $s_\alpha(\alpha+\sigma\alpha) = \sigma\alpha-2\alpha$ is a root of Σ. On the other hand, it follows that $\langle\sigma\alpha-2\alpha, \sigma(\sigma\alpha-2\alpha)\rangle = -\frac{3}{2}\langle\alpha, \alpha\rangle < 0$. This combined with the regularity of (Σ, σ) shows that $\sigma(\sigma\alpha-2\alpha) = -(\sigma\alpha-2\alpha)$. Then $\sigma\alpha = -\alpha$. This contradicts the assumption $\alpha+\sigma\alpha \in \Sigma$. q.e.d.

From now on, we are going to determine all the involutions σ on an irreducible, reduced root system Σ such that (Σ, σ) is an R-root system. As was already remarked, we restrict our attention to such an R-root system (Σ, σ) that (Σ, σ) is not an N-root system. The subsequent discussions are based on Araki's one [Ar]. Namely, first investigate such an involution σ on a root system Σ that $\Sigma(\sigma)$ is of rank 1. Next do the general case for the types B_ℓ, C_ℓ and F_4, separately. (It suffices to discuss on these cases by reason of Lemma 3.1.) Since the discussions are easy but rather long, we only give here the results. We use the notation in [LIE] without notice.

<u>The case of type</u> B_ℓ.

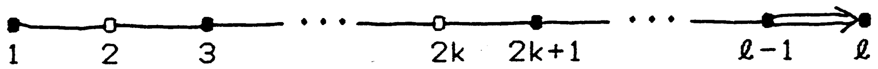
Let (Σ, σ) be an R-root system such that Σ is of type B_ℓ. Let α_i $(1 \leqq i \leqq \ell)$ be a σ-fundamental system of Σ such that



Then there exists a number $k \leqq \ell$ such that

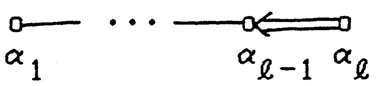
$$\sigma\alpha_{2i-1} = \alpha_{2i-1} \quad (i < k)$$
$$\sigma\alpha_{2i} = -\alpha_{2i-1}-\alpha_{2i}-\alpha_{2i+1} \quad (i < k-1)$$
$$\sigma\alpha_{2k} = -\alpha_{2k-1}-2\alpha_{2k}-\cdots-2\alpha_\ell$$
$$\sigma\alpha_i = \alpha_i \quad (2k < i).$$

Its Satake diagram is given as follows:



<u>The case of type</u> C_ℓ.

Let (Σ, σ) be an R-root system such that Σ is of type C_ℓ. Let α_i $(1 \leqq i \leqq \ell)$ be a σ-fundamental system of Σ such that

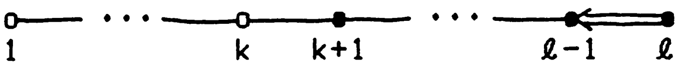


Then we find that there is a number k $(1 < k < \ell)$ such that the involution σ is given by

$$\sigma\alpha_i = -\alpha_i \quad (1 \leqq i < k)$$
$$\sigma\alpha_k = -\alpha_k-2\alpha_{k+1}-\cdots-2\alpha_{\ell-1}-\alpha_\ell$$

$$\sigma\alpha_i = \alpha_i \quad (k < i \leqq \ell).$$

In this case, its Satake diagram is given by



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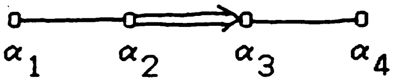
<u>The case of type</u> F_4.

Let (Σ, σ) be an R-root system such that Σ is of type F_4. Let α_i $(1 \leqq i \leqq 4)$ be a σ-fundamental system of Σ such that



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Then σ coincides with one of the involutions defined below:

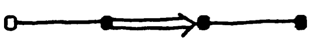
(I) $\sigma\alpha_1 = -\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4$, $\quad \sigma\alpha_i = \alpha_i \ (1 < i \leqq 4)$.

(II) $\sigma\alpha_1 = -\alpha_1-2\alpha_2-2\alpha_3$

$\sigma\alpha_i = \alpha_i \quad (i = 2, 3)$

$\sigma\alpha_4 = -\alpha_2-2\alpha_3-\alpha_4$.

The Satake diagrams of the former and the latter are given by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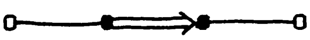


respectively.

<u>Remark 3.2</u>. Let (Σ, σ) be an R-root system but not an N-root system. Assume that Σ is irreducible and reduced. Put

$\Sigma^{\vee} = \{ \frac{2\alpha}{(\alpha,\alpha)}; \alpha \in \Sigma\}$. Then it is known (cf. [LIE]) that $\Sigma^{\vee}$ is also a root system, irreducible and reduced. Furthermore, σ leaves $\Sigma^{\vee}$ invariant. By the above classification, we find that in almost all cases, $(\Sigma^{\vee}, \sigma)$ is an N-root system. The only exceptional case is the involution for the root system of type F_4 defined in (II).

In the above discussions, we always assumed that Σ is reduced. Now we treat a non-reduced one. Hence, let Σ be an irreducble, non-reduced root system. Then it follows ([LIE]) that the type of Σ is BC_{ℓ}. Put $\Sigma_{(s)} = \{\alpha \in \Sigma; \frac{1}{2}\alpha \notin \Sigma\}$ and $\Sigma_{(1)} = \{\alpha \in \Sigma; 2\alpha \notin \Sigma\}$. Then, by definition, $\Sigma_{(s)}$ is of type B_{ℓ} and $\Sigma_{(1)}$ is of type C_{ℓ}.

Let σ be an involution on Σ such that (Σ, σ) is an R-root system. Then σ leaves both of $\Sigma_{(s)}$ and $\Sigma_{(1)}$ invariant. It is clear that one of the following condition holds.

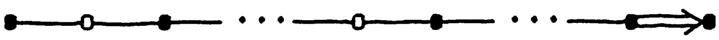
(C.1) $(\Sigma_{(s)}, \sigma)$ is an N-root system.

(C.2) $(\Sigma_{(s)}, \sigma)$ is an R-root system but not an $_{\sigma}$N-root system.

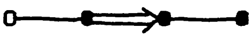
In the case (C.1), $(\Sigma_{(1)}, \sigma)$ is an R-root system but not an N-root system. On the other hand, in the case (C.2), $(\Sigma_{(1)}, \sigma)$ is an N-root system. Noting this, we find that the classifiction of R-root systems related to the root system of type BC_{ℓ} is reduced to the known cases.

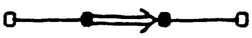
We collect here all the R-root systems (Σ, σ). By the reason given before, it suffices to consider such a one that (Σ, σ) is not an N-root system and that Σ is irreducible.

Result I. Let (Σ, σ) is an R-root system. Assume that Σ is irreducible and reduced and that (Σ, σ) is not an N-root system. Let Ψ be a σ-fundamental system for Σ. Then the Satake diagram for (Σ, Ψ, σ) is one of the followings.

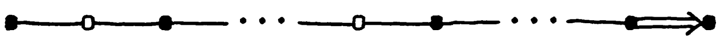
(1) 

(2) 

(3) 

(4) 

Result II. Let (Σ, σ) be an R-root system. Assume that Σ is irreducible and non-reduced and that (Σ, σ) not an N-root system. Let Ψ be a σ-fundamental system for Σ. Then the Satake diagram for (Σ, Ψ, σ) is one of the followings.

(5) 

(6) 

Remark 3.3. Let (Σ, σ) be a root system with an involution. Then we observe the following by using the classification.

In the case where (Σ, σ) is an R-root system, the Satake diagram $S(\Sigma, \Phi, \sigma)$ does not depend on the choice of the

σ-fundamental system Φ.

Let $\underline{g}$ be a real semisimple Lie algebra ane let σ be its involution. Then there exists a Cartan involution θ of $\underline{g}$ commuting with $\theta\sigma = \sigma\theta$. Let $\underline{g} = \underline{h} + \underline{q}$ be the direct sum decomposition for σ and let $\underline{g} = \underline{k} + \underline{p}$ be the Cartan involution for θ. Let $\underline{a}$ be a maximal abelian subspace of $\underline{p} \cap \underline{q}$ and let $\underline{j}$ be a maximal abelian aubspace of $\underline{q}$ containing $\underline{a}$. Let $\underline{j}_c$ be the complexification of $\underline{j}_c$ and let Σ be the root system of $(\underline{g}_c, \underline{j}_c)$. Then it is clear that Σ is regarded as a root system on the real vector space $\underline{j}_0 = \sqrt{-1}(\underline{k} \cap \underline{j}) + \underline{p} \cap \underline{j}$. Put $\sigma = \theta | \underline{j}_0$. Then it follows that (Σ, σ) is an R-root system.

<u>Definition 3.4</u>. An R-root system (Σ, σ) obtained in the above way is called an RE-root system.

By the classification of semisimple symmetric pairs, we find that the R-root systems given in (1)-(6) except (3) are really RE-root systems. For example, we have the following (cf. [OS]).

(1) $(so^*(2n), so^*(2p)+so^*(2n-2p))$ $(2p < n)$

(2) $(su(n+2k, n), su(n+k)+su(n,k)+\sqrt{-1}\mathbb{R})$

(4) $(e_{6(-14)}, su(5,1)+sl(2,\mathbb{R}))$

(5) $(su(p,q), su(k,l)+su(p-k,q-k)+\sqrt{-1}\mathbb{R})$

(6) $(su^*(2p,2q), su^*(2k,2l)+su^*(2(p-k),2(q-l))+\mathbb{R})$

References

[Ar] S. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. of Math., Osaka City Univ., 13 (1962) 1-23.

[LIE] N. Bourbaki, Groupes et algebres de Lie, Chap. IV-VI.

[OS] T. Oshima and J. Sekiguchi, The restricted root system of a semisimple symmetric pair, preprint.

[S] M. Sugiura, Classifications of root systems with involutions and real simple Lie algebras, preprint.

[W] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups.