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Irreducible decompositions of Fock representations of the Virasoro algebra

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Abstract: Explicit irreducible decompositions of Fock representations of the Virasoro algebra are calculated. Each highest weight vector appearing in the decomposition is the character polynomial of general linear group.

0. Introduction.

In this note we give the explicit irreducible decompositions of some highest weight modules of the Virasoro algebra. Highest weight modules of the Virasoro algebra are studied by several authors ([1,2,4,5,8,9,...]). But there are only few results about the irreducible decompositions of the reducible modules. We will decompose the Fock representations (see below) and show that each highest weight vector appearing in the decomposition is nothing but the character polynomial of general linear group. We finally mention about the relationship with some non-linear differential equations of soliton type.

1. The Virasoro algebra and its Fock representations.

First we define the Virasoro algebra (or the string algebra) g as a complex Lie algebra with basis $\{\ell_k\}_{k\in Z}$ and c with the following bracket relations:

$$[\ell_k, \ell_l] = (k-l)\ell_{k+l} + \frac{1}{12}(k^3-k)\delta_{k+l,0}\, c\,, \quad [\ell_k, c] = 0$$

for all integers l,k. Let $N(\xi,\eta)$ be a highest weight g-module, namely, there exists a non-zero vector $v_0 \in N(\xi,\eta)$ such that $\ell_0 v_0 = \xi v_0$, $cv_0 = \eta v_0$, and $\ell_k v_0 = 0$ for $k \geq 1$, where ξ and η are real constants. For each pair (ξ,η) there is a unique g-module $M(\xi,\eta)$ such that any g-module with highest weight (ξ,η) is a quotient of $M(\xi,\eta)$, and there is a unique irreducible highest weight g-module $L(\xi,\eta)$. The module $N(\xi,\eta)$ is the linear span of the elements $\ell_{k_1}\cdots\ell_{k_\nu}v_0$ where $k_1,\ldots,k_\nu$ are non-positive integers. Let $N(\xi,\eta)_m$ be the linear span of such elements with $k_1+\ldots+k_\nu = -m$ for a non-negative intger m. We define the formal character of $N(\xi,\eta)$ by

$$\mathrm{ch}\, N(\xi,\eta) = \sum_{m=0}^{\infty} \dim N(\xi,\eta)_m\, q^m$$

where q is an indeterminate.

We realize a g-module $N(\xi,1)$ as the "Fock representation". Let $V = \mathbb{C}[x_1,x_2,x_3,\ldots]$ be the space of polynomials in infinitely many variables $\{x_j\}_{j=1}^{\infty}$. Put $a_j = \partial/\partial x_j$, $a_{-j} = jx_j$ for $j \geq 1$, and $a_0 = \mu$ where μ is a real constant. For $k \in Z$ we define

$L_k^{(\mu)} = \frac{1}{2}\Sigma_{j\in Z} :a_j a_{k-j}:$ where $:a_i a_j: = \begin{cases} a_i a_j & \text{if } i \leqq j \\ a_j a_i & \text{if } i > j \end{cases}$ is the normal product. Then, by an easy calculation, one can verify that $[L_k^{(\mu)}, L_l^{(\mu)}] = (k-l)L_{k+l}^{(\mu)} + \frac{1}{12}(k^3-k)\delta_{k+l,0}$. The Virasoro algebra g acts on V by $\ell_k \longmapsto L_k^{(\mu)}$ and $c \longmapsto 1$ (the identity operator). We denote this representation by (π_μ, V). It has a highest weight vector (a vacuum vector) $1 \in V$. If π_μ is irreducible, then $V \cong L(\mu^2/2, 1)$, because $\pi_\mu(\ell_0)1 = L_0^{(\mu)}1 = \mu^2/2$.

Proposition 1 ([1,4]). The representation (π_μ, V) is irreducible if and only if $\mu \notin \frac{1}{\sqrt{2}}Z$. If $\mu \in \frac{1}{\sqrt{2}}Z$, then (π_μ, V) is completely reducible.

We study the irreducible decomposition of (π_μ, V) in the case $\mu \in \frac{1}{\sqrt{2}}Z$.

Proposition 2 ([4]). For a non-negative integer n, $\mathrm{ch}\, L(n^2/4, 1) = (1-q^{n+1})\phi(q)^{-1}$ where $\phi(q) = \Pi_{k=1}^{\infty}(1-q^k)$.

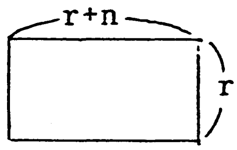
2. Irreducible decomposition of $(\pi_{n/\sqrt{2}}, V)$.

Following to Sato [7], we define the character polynomial $\chi_Y(x)$ for a Young diagram Y as follows:

$$\chi_Y(x) = \Sigma_{\nu_1+2\nu_2+\ldots=N} \; \pi_Y(1^{\nu_1}2^{\nu_2}\ldots)\frac{x_1^{\nu_1}x_2^{\nu_2}\ldots}{\nu_1!\nu_2!\ldots}$$

where N is the size of Y, and $\pi_Y(1^{\nu_1}2^{\nu_2}\ldots)$ denotes the irreducible character of the symmetric permutation group of N letters corresponding to the Young diagram Y and to the conjugacy class consisting ν_1 cycles of size 1, ν_2 cycles of size 2, etc.. For a character polynomial $\chi_Y(x)$, let us denote $\chi'_Y(x) = \chi_Y(\sqrt{2}x)$.

Theorem 3. Fix a non-negative intger n and put $\mu = -n/\sqrt{2}$. Then, for any positive integer k, $L^{(\mu)}_k \chi'_Y(x) = 0$, and $L^{(\mu)}_0 \chi'_Y(x) = \frac{(n+2r)^2}{4} \chi'_Y(x)$ for the Young diagram

Y = 

In the other words, the character polynomial $\chi'_Y(x)$ corresponding to the above Young diagram Y is a highest weight vector of weight $(n+2r)^2/4$.

By calculating the formal character of $\oplus_{r=0}^{\infty} L((n+2r)^2/4,\ 1)$ we get the following corollary.

Corollary 4. $(\pi_{-n/\sqrt{2}},\ V) \simeq \oplus_{r=0}^{\infty} L((n+2r)^2/4,\ 1)$.

3. Hirota's bilinear equations of n-th modified KP-hierarchy.

For a polynomial $f(x) = f(x_1, x_2, x_3, \ldots)$, we denote $\widetilde{f}(D) = f(D_1/\sqrt{2}, D_2/2\sqrt{2}, D_3/3\sqrt{2}, \ldots)$.

Theorem 5. If $f(x)$ is an element of the lower components $\oplus_{r=1}^{\infty} L((n+2r)^2/4,1)$, then $\widetilde{f}(D)$ is a Hirota's bilinear operator of n-th modified KP-hierarchy. (cf. [3,6].)

Examples. a) $\widetilde{\chi'_{\square\square}}(D) = \frac{1}{2}(D_1{}^2+D_2)$ is the 2-weighted homogeneous leading bilinear operator of 1st modified KP-hierarchy.

b) $(L_{-1}^{(-1/\sqrt{2})}\ \chi'_{\square\square})^{\sim}(D) = -\frac{1}{4}(D_1{}^3-3D_1D_2-4D_3)$ is the 3-weighted homogeneous bilinear operator of 1st modified KP-hierarchy.

c) $\widetilde{\chi'_{\square\square\square}}(D) = \frac{1}{6}(D_1{}^3+3D_1D_2+2D_3)$ is the 3-weighted homogeneous leading bilinear operator of 2nd modified KP-hierarchy.

c) $\widetilde{\chi'_{\boxplus}}(D) = \frac{1}{12}(D_1{}^4-4D_1D_3+3D_2{}^2)$ is the leading bilinear operator of the KP-hierarchy.

The number of k-weighted homogeneous bilinear equations of n-th modified KP-hierarchy is calculated and is equal to $p(k-n-1)$ where $p(m)$ is the number of partitions of m.

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