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Automorphic forms, L-functions, and periods integrals.

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Introduction.

Organizers of this symposium asked me to give an expository talk on the representation-theoretic aspects of the arithmetic theory of automorphic forms, especially to give an overview on the works of Langlands. But the author has little experiences in representation theory and the works of Langlands are manifold. So I decided to restrict the talk to some topics especially related the theory of periods of modular forms, which is the center of the current interest of the author. Therefore the talk will be a bit biased to this direction.

In this note, I wrote an introductory part of my talk. I want to discuss some problems on the Hodge structures attached automorphic forms which belong to the representations of the discrete series. These problems would naturally arise, if one want to deepen the Matsushima-Murakami theory and its development, the theory of continuous cohomology (cf. [B-W]).

As the title indicate, the whole talk is motivated by the paper of Deligne:

Valeurs de fonctions L et periodes d'integrales.

Proc. sym. pure Math. 33, Part II, 313-346 (1979).

§1. Definitions of automorphic forms.

In the classical language, the most general definition of automorphic forms might be the following (cf. Harish-Chandra [H]).

Let G be a connected real semisimple Lie group with finite center, and let $\mathfrak{z}$ be the center of the universal envelopping algebra of the complexified Lie algebra $\mathfrak{g}_{\mathbf{C}}$ of G. Choose a maximal compact subgroup K of G and an irreducible finite dimensional representation $\sigma : K \longrightarrow GL(V)$ on the complex vector space V. Let $\lambda : \mathfrak{z} \longrightarrow \mathbf{C}$ be a $\mathbf{C}$-algebra homomorphism. Suppose that Γ is a discreste subgroup of G such that the quotient G/Γ has a finite volume with respect to the Haar measure of G.

<u>Definition</u> (1.1). <u>A modular form</u> f(g) <u>on</u> G (with values in V) <u>of type</u> (σ,λ) with respect to Γ is a C^{∞}-function f(g) on G with values in V, slowly increasing, which satisfies the conditions:

(i) $f(gk)=\sigma(k)f$ for any $g \in G$ and $k \in K$.

(ii) $Xf=\lambda(X)f$ for any $X \in \mathfrak{z}$.

(iii) $f(\gamma g)=f(g)$ for any $\gamma \in \Gamma$.

It is important to discuss this general type of automorphic forms. However, related with algebraic geometry and Diophantine geometry, the most interesting special case seems to be the case where the quotient space G/K is a Hermitian symmetric space, and the mapping λ corresponds to an infinitestimal character of the representation of the discrete series.

<u>Remark</u>. But one has to note that even when G/K is not Hermitian symmetric, we still may have "geometric automorphic forms", geometric in the sense that it should correspond to the L-function of some ℓ-adic representation of Galois group defined by etale

cohomology groups (e.g. recall the theory of base change where $GL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$ are concerned in some case, also recall the Grossen-character of A_0-type on GL(1).)

Let us recall the Borel-Weil type theorem of Narasimahn-Okamoto [N-O], in order to define automorphic forms which belong to the representation of discrete series. So assume that $X=G/K$ is isomorphic to a bounded symmetric domain. Let $\mathfrak{g}=\mathfrak{p}+\mathfrak{k}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and let $C=\mathrm{Ad}(\iota)$ $(\iota \in K)$ define the complex structure on $\mathfrak{p}$. Let $\mathfrak{p}^+$ and $\mathfrak{p}^-$ be the eigenspaces with respect to the operator C with eigenvalues $+i$ and $-i$, respectively. Put $P^+=\exp(\mathfrak{p}^+)$ and $P^-=\exp(\mathfrak{p}^-)$. Then $K=G\cap K_{\mathbb{C}}P^-$. Put $U=K_{\mathbb{C}}P^-$. Then the canonical mapping

$$i:X=G/K \hookrightarrow G_{\mathbb{C}}/U$$

gives the Borel embedding.

Let $\sigma:K \longrightarrow GL(V)$ be a finite dimensional irreducible representation of K on V, and let us denote by the same symbol σ, its extension to a holomorphic finite dimensional extension of the complexification $K_{\mathbb{C}}$ of K. Extending this representation trivially to P^-, we have a finite dimensional irreducible representation of U on V. Associated to this representation, we can define a holomorphic vector bundle F_σ^c on $G_{\mathbb{C}}/U$ (or the corresponding locally free sheaf on $G_{\mathbb{C}}/U$), which has a natural action of $G_{\mathbb{C}}$(hence the corresponding sheaf has $G_{\mathbb{C}}$-linearization). Restricting this to $X=G/K$, we have a holomorphic vector bundle F_σ with G-action compatible that of G on X (or a G-linearized locally free sheaf F_σ on X). Hence the square-integrable $\bar{\partial}$-cohomology groups $H^i_{\bar{\partial},2}(X,F_\sigma)$ has the natural action of G.

Then Narasimahn-Okamoto [N-O] showed that, when the highest weight of σ is (roughly speaking) in a sufficiently general position, these cohomology groups vanish except for one $i=q(\sigma)$, and the survived one gives an irreducible unitary representation of discrete series. Also most of the unitary representations of discrete series is obtained in this way, when G/K is Hermitian.

Let us recall the classical definition of holomorphic automorphic forms. Geometrically speaking, automorphic forms are the sections $\Gamma(\Gamma\backslash X, F_{\sigma,\Gamma})$ of $F_{\sigma,\Gamma}$ on $\Gamma\backslash X$. Here $F_{\sigma,\Gamma}$ is the descent of F_σ with respect to $\pi: X \longrightarrow \Gamma\backslash X$, which is well-defined when the G-linearization of F_σ is induced from a G^{ad}-linearization (In general, the quotient $F_{\sigma,\Gamma}$ might not exist for some Γ). Usually, by using the trivialization $F_\sigma \cong X \times V$ of F on X, this is identified with holomorphic functions on X with values in $V \cong \mathbb{C}^N$ satisfying the automorphy property corresponding to F . $(N=\dim_{\mathbb{C}} V)$.

As an immediate generalization of this, we consider $H^i(\Gamma\backslash X, F_{\sigma,\Gamma})$. Assume that $\Gamma\backslash X$ is smooth and compact. Then

$$H^i(\Gamma\backslash X, F_{\sigma,\Gamma}) = \mathrm{Hom}_{\mathbb{C}}(H^{n-i}(\Gamma\backslash X, F^\vee_{\sigma,\Gamma} \otimes \Omega^n_{\Gamma\backslash X}), \mathbb{C})$$

by Serre duality. Here $n=\dim_{\mathbb{C}} X$ and $\Omega^n_{\Gamma\backslash X}$ is the canonical bundle or canonical sheaf of $\Gamma\backslash X$. $F^\vee_{\sigma,\Gamma}$ is the dual of $F_{\sigma,\Gamma}$.

Let us consider the right regular representation of G on $L^2(\Gamma\backslash G)$. Then the Frobenius reciprocity implies that

$$\mathrm{Hom}_{\mathbb{C}}(H^{n-i}(\Gamma\backslash X, F^\vee_{\sigma,\Gamma} \otimes \Omega^n_{\Gamma\backslash X}), \mathbb{C}) = \mathrm{Hom}_G(H^{n-i}_{\bar\partial,2}(X, F^\vee_\sigma \otimes \Omega^n_X), L^2(\Gamma\backslash G)).$$

Thus the space of automorphic cohomology group $H^i(\Gamma\backslash X, F_{\sigma,\Gamma})$ is identified with the intertwing space

$$\mathrm{Hom}_G(H^{n-i}_{\bar\partial,2}(X, F^\vee_\sigma \otimes \Omega^n_X), L^2(\Gamma\backslash G)).$$

Assume that σ is in a sufficiently general position, and let λ be the infinitestimal character corresponding to the representation of discrete series of G on $H^{n-i}_{\bar\partial,2}(X,F^{\vee}_{\sigma}\otimes\Omega^n_X)$. Let $\varphi(v)$ be the evaluation of an element of $\mathrm{Hom}_G(H^{n-i}_{\bar\partial,2}(X,F^{\vee}_{\sigma}\otimes\Omega^n_X),L^2(\Gamma\backslash G))$ at a vector v of $H^{n-i}_{\bar\partial,2}(X,F^{\vee}_{\sigma}\otimes\Omega^n_X)$. If $\varphi(v)(g)$ is a C^{∞}-function, then we have $X\varphi(v)=\lambda(X)\varphi(v)$ for any $X\in\mathfrak{z}$. Moreover, consider the K-type decomposition of the representation $H^{n-i}_{\bar\partial,2}(X,F^{\vee}_{\sigma}\otimes\Omega^n_X)$, which is described by the Blattner-Hecht-Schmid theorem ([ ],[ ],[ ]) Say, consider the lowest K-type (ρ ,W) of this representation, which occurs with multiplicity one. Let $W^{\vee}$ be the dual space of W, and $\rho^{\vee}$ the contragradient representation of ρ. Then, evaluating vectors of $\mathrm{Hom}_G(H^{n-i}_{\bar\partial,2}(X,F^{\vee}_{\sigma}\otimes\Omega^n_X),L^2(\Gamma\backslash G))$ at the vectors in W, we have

$$\mathrm{Hom}_G(H^{n-i}_{\bar\partial,2}(X,F^{\vee}_{\sigma}\otimes\Omega^n_X),L^2(\Gamma\backslash G)) \hookrightarrow A(\Gamma;(\rho^{\vee},\lambda)).$$

Here $A(\Gamma;(\rho^{\vee},\lambda))$ is the space of automorphic forms of type $(\rho^{\vee},\lambda)$ in the sense of definition (1.1).

This fact justifies the following.

<u>Definition</u> (1.2) Let X, Γ, $F_{\sigma,\Gamma}$ be as above. Then we say that the elements of the space $H^i(\Gamma\backslash X,F_{\sigma,\Gamma})$ are <u>automorphic forms of discrete series with respect to</u> Γ <u>of type</u> (σ,i).

<u>Remark.</u> If we replace, the lowest K-type (ρ,W) by another K-type which might not be lowest, we have also obtain the elements of $A(\Gamma;\tau,\lambda)$ for some representation τ of K. These modular forms are obtained from those of Definition (1.2) applying the elements of $U(\mathfrak{g}_{\mathbb{C}})$. When i=0, we have non-holomorphic modular forms from holomorphic one f(g) applying $S(\mathfrak{g}^{+}_{\mathbb{C}})$ to f (Note $X^{-}f=0$ for $X^{-}\in\mathfrak{p}^{-}_{\mathbb{C}}$).

These modular forms are discussed by Shimura (e.g. [ ]) and Harris (e.g. [ ]).

From the viewpoint of analytic number theory, the most interesting problem is to consider L-functions attached to them. Hecke first considered this problem systematically for elliptic modular forms, probably investigating the work of Hurwitz. Among others, he showed that the L-functions attached to those modular forms which are eigenforms of all Hecke operators have Euler products, Hecke operators being certain linear operators defined over the space of modular forms.

Let us explain Hecke operators briefly. For simplicity, we assume that our Lie group G is the real points of an algebraic group $\underset{\sim}{G}$ defined over $\mathbb{Q}$, which has a faithful rational representation $R:\underset{\sim}{G} \longrightarrow GL(N)$ defined over $\mathbb{Q}$. Let $\Gamma=\underset{\sim}{G}_{\mathbb{Z}}=R^{-1}(GL(N;\mathbb{Z}))$. Then Γ is an arithmetic discrete subgroup of $G=G(\mathbb{R})$. In general, two subgroups Γ_1 and Γ_2 of G are called commensurable, if the indices $[\Gamma_i:\Gamma_1\cap\Gamma_2]$ $(i=1,2)$ are both finite. For a fixed subgroup Γ of G, the elements δ of G such that Γ and $\delta\Gamma\delta^{-1}$ are commensurable make a group containing Γ in G. This group is called the commensurator of Γ in G. In our case, $\Delta=\underset{\sim}{G}(\mathbb{Q})=R^{-1}(GL(N;\mathbb{Q}))$ is contained in the commensurator of $\Gamma=\underset{\sim}{G}(\mathbb{Z})$.

Let $\alpha\in\Delta$. Then the double coset $\Gamma\alpha\Gamma$ in Δ is written as a finite (disjoint) union $T_\alpha=\Gamma\alpha\Gamma=\bigcup_{i=1}^{m}\beta_i\Gamma$ of right cosets. Let $f(g)$ be an element of $A(\Gamma;\sigma,\lambda)$ for some $\sigma:K\longrightarrow GL(V)$ and $\lambda:\mathfrak{z}\longrightarrow\mathbb{C}$. Then we can check that $T_\alpha f$ defined by

$$(T_\alpha f)(g)=\sum_{i=1}^{m} f(\beta_i g)$$

also belongs to $A(\Gamma;\sigma,\lambda)$.

In general, for any function ψ on the double coset space $\Gamma\backslash\Delta/\Gamma=G(\mathbb{Z})\backslash G(\mathbb{Q})/G(\mathbb{Z})$ with finite support, we can define $T_\psi f$ by

$$T_\psi f=\sum_{\Gamma\alpha\Gamma\in \mathrm{Supp}(\psi)} \psi(\Gamma\alpha\Gamma)T_\alpha f,$$

where the summation is taken over those finite cosets such that $\psi(\Gamma\alpha\Gamma)\neq 0$. Let us call these operators T_ψ the Hecke operators, if T_ψ is $\mathbb{Z}$-valued. If f(g) is an eigenform of T_ψ, then $T_\psi f=a_\psi f$ with the eigenvalue $a_\psi\in\mathbb{C}$ of f with respect to T_ψ.

Let $C_0(G(\mathbb{Z})\backslash G(\mathbb{Q})/G(\mathbb{Z}))$ be the space of functions on the double coset space $G(\mathbb{Z})\backslash G(\mathbb{Q})/G(\mathbb{Z})$ with finite supports. Then, roughly speaking, we have a tensor product decomposition

$$C_0(G(\mathbb{Z})\backslash G(\mathbb{Q})/G(\mathbb{Z}))\cong \bigotimes_{p\ \text{primes}} C_0(G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)),$$

where p is taken over all primes, and $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are p-adic integers and p-adic numbers, respectively. Here the infinite tensor product of the right hand side of the above isomorphism is the "restricted tensor product" in the sense that any element is of the form $\psi=\bigotimes_p \psi_p$ with $\psi_p=X_{G(\mathbb{Z}_p)}$=the characteristic function of $G(\mathbb{Z}_p)$ for almost all p (i.e. except for a finite number of primes). Note that for almost all p, $G(\mathbb{Z}_p)$ is a maximal compact subgroup of $G(\mathbb{Q}_p)$.

<u>Remark</u>. We discuss here the typical situation. Actually, we have to consider the "bad" primes of G more carefully.

Therefore, via this isomorphism, we consider Hecke operator T_{α_p} with non-trivial component only at unique p:

$$T_{\alpha_p}=\Gamma\alpha_p\Gamma=\bigotimes_{q\ \text{primes}}\psi_q \quad (\psi_q=X_{G(\mathbb{Z}_q)}\ \text{if}\ p\neq q).$$

THus, if f(g) is an eigenform of all Hecke operators, it defines a homomorphism

$$\begin{array}{ccc} C_0(\underset{\sim}{G}(\mathbb{Z}_p)\backslash \underset{\sim}{G}(\mathbb{Q}_p)/\underset{\sim}{G}(\mathbb{Z}_p)) & \longrightarrow & \mathbb{C} \\ \Psi & & \Psi \\ T_{\alpha_p}=\Gamma\alpha_p\Gamma & \longmapsto & a_p, \end{array}$$

by

where a_p is the eigenvalue of f with respect to T_{α_p}: $T_{\alpha_p} f=a_p f$. Therefore, if $\underset{\sim}{G}(\mathbb{Z}_p)$ is a maximal compact subgroup of $\underset{\sim}{G}(\mathbb{Q}_p)$, an eigenform of all Hecke operators defines a spherical measure on $\underset{\sim}{G}(\mathbb{Q}_p)$ for each p. This measure, in turn, defines a representation of class 1 on $\underset{\sim}{G}(\mathbb{Q}_p)$ for such p.

A more direct way to attach a representation to an eigenform is the following.

Let us consider the set I of all discrete subgroup Γ' of G commensurable with $\Gamma=\underset{\sim}{G}(\mathbb{Z})$, which contain a cogruence sungroup of Γ (N.B. this is always true, if the congruence subgroup problem is true for G). This set I is ordered by

$$\Gamma_1 \leq \Gamma_2 \quad \Longleftrightarrow \quad \Gamma_1 \supseteq \Gamma_2.$$

The set I is obviously a directed set. Let us consider the inductive limit

$$\tilde{L}^2(\underset{\sim}{G}(\mathbb{Q});G):=\varinjlim_{\Gamma_i \in I} L^2(\Gamma_i\backslash G),$$

where the morphism $L^2(\Gamma_1\backslash G) \longrightarrow L^2(\Gamma_2\backslash G)$ of the inductive system for $\Gamma_1\leq\Gamma_2$, is the canonical mapping defined by the pull-back. Obviouly $\Delta=G(Q)$ acts on this space by

$$\delta: f(g) \in L^2(\Gamma\backslash G) \longmapsto f(\delta g) \in L^2(\delta\Gamma\delta^{-1}\backslash G).$$

Also, for each element $v \in L^2(\underset{\sim}{G}(\mathbb{Q});G)$, there exists a congruence subgroup Γ' of Γ such that Γ' acts on v trivially. Therefore, the profinite completion $\hat{\Gamma}=\varprojlim_{\substack{\Gamma' \text{ a congruence} \\ \text{subgroup of } \Gamma}} \Gamma/\Gamma' =\prod_p \underset{\sim}{G}(\mathbb{Z}_p)$ acts on the space $\tilde{L}^2(\underset{\sim}{G}(\mathbb{Q});G)$. Hence the product $\hat{\Gamma}\cdot\Delta=\underset{\sim}{G}(\mathbb{Q})\prod_p \underset{\sim}{G}(\mathbb{Z}_p)$ acts on $\tilde{L}^2(G(\mathbb{Q});G)$. The last group gives the finite part G_0 of the adele group $G_{\mathbb{A}}$, if the strong approximation is true for G. In general it defines an index-finite subgroup of G_0.

Moreover, we can consider the action of the universal envelopping algebra $U(\mathfrak{g}_{\mathbb{C}})$ on the C^∞-vectors $\tilde{L}^2(\Delta;G)$ $=\varinjlim_{\Gamma_i\in I} L^2(\Gamma_i\backslash G)$. Thus we can reagrd $\tilde{L}^2(\Delta;G)$ as $G(\mathbb{Q})\prod_p G(\mathbb{Z}_p)\times U(\mathfrak{g}_{\mathbb{C}})$ module. And by completion, $\tilde{L}^2(\Delta;G)$ as a $\underset{\sim}{G}(\mathbb{Q})\cdot\prod_p \underset{\sim}{G}(\mathbb{Z}_p)\times \underset{\sim}{G}(\mathbb{R})$ module. The group $\underset{\sim}{G}(\mathbb{Q})\cdot\prod_p \underset{\sim}{G}(\mathbb{Z}_p)\times \underset{\sim}{G}(\mathbb{R})$ is the adele group $\underset{\sim}{G}_{\mathbb{A}}$, if the strong approximation is valid for G. In general, it is an index finte subgroup of $G_{\mathbb{A}}$. Assume that $\underset{\sim}{G}_{A}=\underset{\sim}{G}(\mathbb{Q})\prod_p \underset{\sim}{G}(\mathbb{Z}_p)\underset{\sim}{G}(\mathbb{R})$, for simplicity. Then the adele group $G_{\mathbb{A}}$ acts on $\tilde{L}^2(G(\mathbb{Q});G)$.

Starting with an automorphic form f(g) in some $A(\Gamma;\sigma,\lambda)$ and in $L^2(\Gamma\backslash G)$, we can consider the closure of a subspace of $\tilde{L}^2(\Delta;G)$ generated by

$$i_{\Gamma'}\{\, Xf(\delta g) \mid X\in U(\mathfrak{g}_{\mathbb{C}});\delta\in\Delta \text{ , such that } \delta\Gamma\delta^{-1}\supset\Gamma' \,\},$$

where $i_{\Gamma'}$ is the canonical mapping

$$i_{\Gamma'}:L^2(\Gamma'\backslash G) \longrightarrow L^2(\Delta;G).$$

Let us denote this subspace of $L^2(G(\mathbb{Q});G)$ by V_f. Then V_f defines a subrepresentation π_f of $G_{\mathbb{A}}$ in $L^2(G(\mathbb{Q});G)$. If f(g) is an automorphic form of discrete series of type (σ,i) in the sense of Definition (1.2), the infinte part $\pi_{f|G(\mathbb{R})}$ of the represenation π_f is the representation of the discrete series, equivalent to the unitary representation $H^{n-i}_{\bar{\partial},2}(X,\check{F}_\sigma\otimes\Omega^n_X)$. And, if f(g) is an eigenform of all Hecke operators on $G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ for some p such that $G(\mathbb{Z}_p)$ is a maximal compact subgroup of $G(\mathbb{Q}_p)$, the p-component $\pi_{f|G(\mathbb{Q}_p)}$ of π_f is a representation of class 1, corresponding to the spherical measure on $G(\mathbb{Q}_p)$ determined by the eigenvalues of f(g) at p. Note that in this case, f itself a $\prod_p G(\mathbb{Z}_p)$-invariant vector in V_f, And also at ∞, f is a vector belonging to the lowest K-type of the representation on the space $H^{n-i}_{\bar{\partial},2}(X, F_\sigma^\vee \otimes \Omega_X^n)$. We can show that f is an eigenform of all Hecke operators, then the representation (π_f, V_f) generated by f in $\tilde{L}^2(G(\mathbb{Q});G)$ is an irreducible representation.

In the classical theory of automorphic forms, holomorphic modular forms which are eigenforms of Hecke operators played a privileged role. But, from the viewpoint of the representation theory, these are vectors which have special $(\prod_p G(\mathbb{Z}_p), K)$-type in the irreducible automorphic representations in

$$\varinjlim_{L \to \{e\}} L^2(G(\mathbb{Q})\backslash G_{\mathbb{A}})^L,$$

where $G_{\mathbb{A}}$ the adelization of G, L open compact subgroups in G_0 the finite part of $G_{\mathbb{A}}$. So it has been one problem to give "intrinsic" definitions for important arithmetic objects (say, L-functions) attached to automorphic forms, in terms of the automorphic representation. Also it has been a problem to rewite classically important methods and results into the setting of the represntation theory, say Fourier expansions as Whittaker models, theta series as Weil representations.

2. Definition of L-functions attached automorphic forms.

Let us review the definition of modular forms attached to automorphic forms in this chapter.

The most general definition of L-functions attached to automorphic eigenforms is due to Langland [L1], and a bit more general one is discussed in [L2], [B]. This is an intermediate step to attach L-functions to automorphic representations. But for good primes p, the definition for automorphic forms suffices.

Let us consider the simplest and typical case discussed in [L1]. Let $\mathfrak{g}$ be a split semisimple Lie algebra over $\mathbb{Q}$, and G its adjoint group. Choose a split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a Chevalley basis of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\mathfrak{g}_{\mathbb{Z}}$ be the lattice generated by the Chevalley basis over $\mathbb{Z}$. Then, for each prime p, let $G_{\mathbb{Z}_p}$ be the stabilizer of $\mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in $G_{\mathbb{Q}_p}$. At infinite place, let $G_{\mathbb{Z}}$ be the maximal compact subgroup of $G_{\mathbb{Q}_\infty} = G_{\mathbb{R}}$ corresponding to the involution associated to the given Chevalley basis.

Put

$$K_0 = \prod_{p \text{ finite}} G_{\mathbb{Z}_p}.$$

Then K_0 is an open compact subgroup of the finite part G_0 of the adelization $G_{\mathbb{A}}$ of G. For any open compact subgroup L of G_0, we consider the space

$$L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})^L$$

of the square-integrable function on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$, which is invariant under L. If $L \subset L'$, there is a canonical mapping

$$L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})^{L'} \hookrightarrow L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})^L.$$

Let us consider the inductive limit

$$\mathcal{L}(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}) = \varinjlim_{L \to \{e\}} L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})^L.$$

Let φ be an element of $\mathcal{L}(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$. Then $\varphi \in L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})^L$ for some L, and $L = \prod_{p \text{ finite}} L_p$, L_p being the p-component of L for each p, which is equal to $G_{\mathbb{Z}_p}$ for almost all p. Assume that φ belongs to the representation space V of an irreducible subrepresentation (π, V) of $G_{\mathbb{A}}$ in $\mathcal{L}(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$. Then $\pi = \bigotimes_{p \text{ all places}} \pi_p$ with irreducible local components π_p.

If $L_p = G_{\mathbb{Z}_p}$ for p, the element φ defines a $G_{\mathbb{Z}_p}$-invariant vector under the representation π_p, hence π_p is of class 1. Let us call such a prime an unramified prime for φ, or good prime for φ.

Let p be a good prime, and let H_p the algebra of compact supported measure on $G_{\mathbb{Q}_p}$, biinvariant under $G_{\mathbb{Z}_p}$. Then let us define the action of $\mu \in H_p$ on φ, by

$$T_\mu(\varphi)(g) = \int_{G_{\mathbb{Q}_p}} \varphi(gh)\mu(h).$$

Then the irreducibility of π_p implies that

$$T_\mu(\varphi) = \lambda_p(\mu)\varphi$$

with a complex number $\lambda_p(\mu) \in \mathbb{C}$.

Thus we have a $\mathbb{C}$-algebra homomorphism

$$\mu \in H_p \longrightarrow \lambda_p(\mu) \in \mathbb{C},$$

obtained by φ for good p.

Let T be the Cartan subgroup of G corresponding to $\mathfrak{f}$. Then, recall that the algebra homomorphisms $H_p \longrightarrow \mathbb{C}$ are parametrized in terms of the homomorphisms $T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p} \longrightarrow \mathbb{C}^\times$. Note that first each algebra homomorphism $H_p \longrightarrow \mathbb{C}$ determines a zonal spherical function ω (dg the Haar measure). And zonal spherical functions are constructed as follows (cf. MacDonald [M]).

Let us fix a Borel subgroup B of G containing T. Then $G_{\mathbb{Q}_p}=B_{\mathbb{Q}_p}\times G_{\mathbb{Z}_p}$ and $B_{\mathbb{Q}_p}\cap G_{\mathbb{Z}_p}=B_{\mathbb{Z}_p}$. Let us consider a homomorphism

$$\chi : T_{\mathbb{Q}_p} \longrightarrow \mathbb{C}^{\times}$$

trivial on $T_{\mathbb{Z}_p}$ (i.e. unramified character on $T_{\mathbb{Q}_p}$).
Extending this trivially across the unipotent part $U_{\mathbb{Q}_p}$ of $B_{\mathbb{Q}_p}$ (U is the unipotent radical of B), we have a continuous homomorphism, again denoted by χ,

$$\chi : B_{\mathbb{Q}_p} \longrightarrow \mathbb{C}^{\times}.$$

Let $\delta : B_{\mathbb{Q}_p} \longrightarrow \mathbb{R}^{\times}$ be the function of module on $B_{\mathbb{Q}_p}$. Then we consider a function $\psi_\chi(g)$ on $G_{\mathbb{Q}_p}$ defined by

$$\psi_\chi(g) := \chi(b)\delta(b)^{\frac{1}{2}} \text{ for the Iwasawa decomposition}$$
$$g=bk \qquad (b\in B_{\mathbb{Q}_p},\quad k\in G_{\mathbb{Z}_p}).$$

Then the integral

$$\omega_\chi(g)=\int_{G_{\mathbb{Z}_p}} \psi_\chi(kg)\,dk$$

defines a zonal spherical function on $G_{\mathbb{Q}_p}$. Then, for any compact supported measure $\mu\in H_p$, we have

$$\mu * \omega_\chi = \lambda(\mu)\,\omega_\chi$$

where $\lambda(\mu)$ is a complex number, and the mapping $\mu\in H_p \longmapsto \lambda(\mu)\in\mathbb{C}$ defines an algebra homomorphism.

Any homomorphism $H_p \longrightarrow \mathbb{C}$ is obtained in this manner. Moreover, we have $\omega_\chi=\omega_{\chi'}$, for any two characters χ, $\chi' : T_{\mathbb{Q}_p} \longrightarrow \mathbb{C}^{\times}$, if and only if there is an element w of the Weyl group $W_G=N(T)/T$ such that $\chi'(t)=\chi(t^w)$ for all $t\in T_{\mathbb{Q}_p}$.

Let Φ be the root system for the pair $(\mathfrak{g},\mathfrak{t})$, and let M be the lattice in $\mathrm{Hom}(\mathfrak{t},\mathbb{R})$ generated by the roots in Φ. For any element $m=\sum_{\alpha\in\Phi} n_\alpha\alpha$ $(n_\alpha\in\mathbb{Z})$ of M, we can attach a character $\xi_m:T\longrightarrow\mathbb{G}_m$ by

$$\xi_m=\sum_{\alpha\in\Phi}\xi_\alpha^{n_\alpha},$$

where ξ_α is the character associated to $\alpha\in\Phi$. Passing to the $\mathbb{Q}_p$-rational points, we have a homomorphism

$$\bar{\xi}_m:T_{\mathbb{Q}_p}\xrightarrow{\ \xi_m\ }\mathbb{G}_{m\mathbb{Q}_p}\cong\mathbb{Q}_p^\times\xrightarrow{\ \mathrm{ord}_p\ }\mathbb{Z}$$

trivial on $T_{\mathbb{Z}_p}$, which gives a perfect pairing

$$\begin{array}{ccc}(T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p})\times M & \longrightarrow & \mathbb{Z}.\\ (t,\ m) & \longmapsto & \bar{\xi}_m.\end{array}$$

Hence we can identify $T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p}$ with $M^\vee=\mathrm{Hom}(M,\mathbb{Z})$. Let us denote this by $\log_p:T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p}\cong M^\vee$.

Let $\alpha^\vee$ be the coroot in $\mathrm{Hom}(\mathfrak{t},\mathbb{R})^\vee=\mathfrak{t}\otimes_{\mathbb{Q}}\mathbb{R}$ the dual space of $\mathrm{Hom}(\mathfrak{t},\mathbb{R})$ corresponding to the root α. And let $\Phi^\vee$ be the dual root system. Let $\mathfrak{g}^\vee$ be the Lie algebra and a split Cartan subalgebra $\mathfrak{t}^\vee$ over $\mathbb{C}$ corresponding to $\Phi^\vee$. Let $N^\vee$ be the lattice in $\mathfrak{t}\otimes_{\mathbb{Q}}\mathbb{R}=\mathrm{Hom}(\mathfrak{t},\mathbb{R})^\vee$ generated by $\alpha^\vee\in\Phi^\vee$. Then $N^\vee\subseteq M^\vee\subset\mathfrak{t}\otimes_{\mathbb{Q}}\mathbb{R}$, by the natural identification. Let $G^\vee$ be the simply connected Lie group with Lie algebra $\mathfrak{g}^\vee$, and let $T^\vee$ be the Cartan subalgebra corresponding to $\mathfrak{t}^\vee$. Then, we have a natural identification

$$\mathfrak{t}^\vee\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathrm{Hom}(M^\vee,\mathbb{R}).$$

Exponetiating this, we have the identification

$$T_{\mathbb{C}}\cong\mathrm{Hom}(M^\vee,\mathbb{C}^\times).$$

Then associated to any homomorphism

$$\chi:T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p}\longrightarrow\mathbb{C}^\times,$$

we have

$$\chi \circ \log_p^{-1} : M \longrightarrow \mathbb{C}^\times,$$

which defines an element of $T^\vee_{\mathbb{C}}$ via the identification $\mathrm{Hom}(M^\vee, \mathbb{C}^\times) \cong T^\vee_{\mathbb{C}}$. Therefore, associated to each algebra homomorphism $\lambda_p : H_p \longrightarrow \mathbb{C}$, there is a W_G-orbit in $T^\vee_{\mathbb{C}}$, or a semisimple conjugacy class $\{g_p\}$ in $G^\vee_{\mathbb{C}}$.

Thus the automorphic form φ determines a homomorphism $\lambda_p : H_p \longrightarrow \mathbb{C}$ for good p, accordingly determines a conjugacy class $\{g_p\}$ in $G^\vee_{\mathbb{C}}$ corresponding to λ_p. Suppose a finite dimensional complex (rational) representation σ of $G^\vee_{\mathbb{C}}$ is given. Then we can consider the Euler product

$$L(s,\varphi,\sigma) = \prod_{p \text{ good}} \det(1-\sigma(g_p)p^{-s})^{-1},$$

which is absolutely convergent for Re s sufficiently large, as shown by Langlands [L], [L]. In [L], he also proved that $L(s,\varphi,\sigma)$ has an analytic continuation to the whole s-plane for various G and some σ. The main idea of proof is to write $L(s,\varphi,\sigma)$ as a quotient $c(s)/c(s+1)$ of the constant terms $c(s)$ of the Eisenstein series $E(\tilde{g}, s, \varphi)$ which "lifts" φ to a bigger group $\tilde{G}$ ($\tilde{g} \in \tilde{G}$) such that G is contained in $\tilde{G}$ as the Levi component of a cuspidal pair in $\tilde{G}$.

3. Hodge structures attached automorphic forms.

The theory of motives is a conjectural theory of cohomology proposed by Grothendieck and Deligne. It expects that there is a universal cohomology theory for algebraic varieties defined over aritymetic fields such that any know cohomology theories (say, etale, Betti, Hodge, etc.) are derived from this by functorial ways. When we consider the cohomology of degree 1, the motives are embodied by abelian varieties. Though it is still a conjectural theory, it has been a productive idea, especially in many important works of Deligne.

So any cohomology-chauvinist will necessarily claim that the first important task in the arithmetic research of automorphic forms is to attach motives to automorphic forms (or automorphic representations). As a corollary of this task (and also as an intermediate step to this task), one might consider a problem to attach Hodge structures to automorphic forms, i.e. investigation of periods of automorphic forms.

The first step to this direction is done by Matsushima-Murakami [M-M]. They developed Hodge theories for the cohomology groups over modular varieties with coefficients in the local systems which are variations of Hodge structures derived from natural fibre spaces over those modular varieties. One missing point in this theory seems to be the characterization of the Hodge components. This is done by Langlands, Borel-Wallach [B-W], by using continuous cohomology to find the relation between Hodge components and the representations of discrete series.

On the other hand, Deligne show that the setting of Matsushima-Murakami is a special case of the smooth variations over smooth projective varieties.

The first arithmetic step is to invesigate the action of Hecke operators on these Hodge structures. In this point, an observation of Hotta-Parthasarathy [H-P] is very important.

I am going to discuss these problems in more details in my talk.

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