ユニタリ表現論セミナー報告集 III, 1983
pp.1-14

Tensoring method for semisimple groups

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§ 0. Introduction

Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}_0$ its Lie algebra. We consider admissible representations of finite length of G. To an irreducible admissible representation E, we can associate a λ which is called infinitesimal character or central character of E. We write this as $E = E_\lambda$.

Let's explain "tensoring method" in large. Let F be a finite dimensional representation of G. Taking the highest or lowest component of $E_\lambda \otimes F$, we can deduce some properties of irreducible representations with "singular" parameters from those of irreducible representations with "regular" parameters. For instance, Bernstein-Gel'fand-Gel'fand [2] used this method to get a property of Verma modules with singular highest weights from that of Verma modules with regular highest weights. And also, G.Zuckerman [11] applied it to get the properties of limits of discrete series representations from those of discrete series representations. Many other people used the method to study admissible representations of G, too (see [1], [3], [7]).

In this lecture, we will report some results about decomposition of $E_\lambda \otimes F$. In § 1, we treat general G and get fairly natural results. In § 2, G is the Lorentz group of 5-th order and we report some interesting examples. In the case that G is the Lorentz group of n-th order, we can decompose $E_\lambda \otimes F$ for any irreducible admissible representation E_λ. But the results are so complicated that we cannot talk about it in this short lecture. Please see [8].

We add the table of decompositions of $E_\lambda \otimes F$ in the case that $G = SL(2,\mathbb{R})$ as Appendix.

§ 1. Decompositions of $E_\lambda \otimes F$ for general G

Let G and $\underline{g}_0$ as in § 0. We denote by $\underline{g}$ the complexification of $\underline{g}_0$ (if H is a Lie group, we always denote its Lie algebra by corresponding small letter subscripted by 0. And the complexification of $\underline{h}_0$ is denoted by $\underline{h}$, eliminating 0.) Fix a maximal compact subgroup K of G and $G = KAN$ be an Iwasawa decomposition of G.

1.1. At first we consider the case that E_λ is a discrete series representation. So, we assume that rank G = rank K. Then, if we choose a Cartan subgroup H of K, H is also a Cartan subgroup of G. We fix this Cartan subgroup H, and

Weyl group, roots, etc. are considered with respect to this pair $(\underline{g}, \underline{h})$.

Let ρ = (half the sum of positive roots) with respect to suitable ordering, and put

$$\Lambda = \{\lambda \in \underline{h}^* \mid \exp \lambda(X) \quad (X \in \underline{h}_0) \text{ defines a character of } H\}$$

or integral weights. If we take a regular element $\lambda \in \Lambda + \rho$, then there is a unique discrete series representation D_λ such that

$$\nabla(C_\lambda)\theta(D_\lambda)(\exp X) = (\pm 1) \sum_{s \in W(H;G)} \det(s) \exp s\lambda(X) \quad (X \in \underline{h}_0),$$

where $\nabla(C_\lambda)$ is Weyl denominator corresponding to the chamber C_λ which contains λ, and $\theta(D_\lambda)$ is the character of D_λ, $W(H;G) = N_G(H)/Z_G(H)$. This correspondence between regular elements of $\Lambda+\rho$ and discrete series representations is bijective modulo the action of $W(H;G)$ on $\Lambda+\rho$ (see [4]).

Let's consider $D_\lambda \otimes F$, where F is a finite dimensional representation of G. We denote by $P(F)$ the set of weights of F not considering multiplicities. For an $\eta \in P(F)$, put

$$\Phi_\eta = \{(s,\nu) \in W \times P(F) \mid \lambda + \eta = s\lambda + \nu\},$$

where W denotes Weyl group of $(\underline{g}, \underline{h})$. Now we state the main result of this paragraph.

Proposition 1.1. Let $\lambda \in \Lambda+\rho$ be regular. Suppose $\lambda+\eta$ is regular for a fixed $\eta \in P(F)$. Then the character is decomposed on the compact Cartan subgroup H as follows:

$$\theta(\mathrm{Proj}(\lambda+\eta)(D_\lambda \otimes F)) = \sum_{i=1}^{n} c_i \ \theta(D_{w_i(\lambda+\eta)}) ,$$

where c_i's are integers and $\mathrm{Proj}(\lambda+\eta)$ denotes the projection to the quasi-simple component which has infinitesimal character $\lambda+\eta$. The integers $\{c_i\}$ are given as follows. Let $\{w_i \mid 1 \leqq i \leqq n\}$ be a complete system of representatives of $W/W(H;G)$ and put

$$\Phi_\eta^i = \{(s,\nu)\in\Phi_\eta \mid s \in w_i^{-1}W(H;G)\} .$$

Then,

$$c_i = \frac{\nabla(C_{w_i(\lambda+\eta)})}{\nabla(C_\lambda)} \det(w_i) \sum_{(s,\nu)\in\Phi_\eta^i} m(\nu)\det(s),$$

where $m(\nu)$ denotes the multiplicity of ν.

Remark. One can see that $c_i \in \mathbb{Z}$ may be negative (see 2.3 (i)').

This proposition doesn't imply the character identity on the whole G. But this formula contains some "information" about $D_\lambda \otimes F$. The author doesn't know what this "information" is for general G. But in the case that G is the Lorentz group, we get

$$\text{(multiplicity of } D_\mu^+ \text{ in } D_\lambda \otimes F)$$
$$- \text{(multiplicity of } D_\mu^- \text{ in } D_\lambda \otimes F)$$

from above proposition ($D_*^{\pm}$ are two series of discrete series representations, and these are all for the Lorentz group, see 2.1).

1.2. In the second, we consider principal series representations. In this paragraph, we don't assume that rank G = rank K.

Let $M = Z_K(A)$ and put $P = MAN$, a minimal parabolic subgroup of G. Take an irreducible representation σ of MA and consider it as an irreducible representation of P trivial on N. Then we call $T^\sigma = \mathrm{Ind}_P^G \sigma$ a (non-unitary) principal series representation induced from σ.

<u>Proposition 1.2.</u> Let $T^\sigma = \mathrm{Ind}_P^G \sigma$ be a principal series representation and F a finite dimensional representation of G. Then there exists a set $\{\sigma_i \mid 1 \leqq i \leqq n\}$ of finite dimensional representations of MA such that

$$\theta(T^\sigma \otimes F) = \sum_{i=1}^{n} \theta(T^{\sigma_i}).$$

This proposition shows that (principal series representation)$\otimes F$ decomposes into the sum of principal series representations. But as we will mention later, T^{σ_i} may be reducible even if T^{σ} is irreducible. One can easily calculate $\{\sigma_i\}$ which are the set of composition factors of $\sigma \otimes (F|_{MA})$.

1.3. Proofs of Propositions 1.1 and 1.2 are similar and easy. For Proposition 1.1, we calculate the character of $D_\lambda \otimes F$ on compact Cartan subgroups and have the result. While in the proof of Proposition 1.2, we work on Cartan subgroups with maximal vector part, and use the explicit formula of the characters of principal series representations.

We mention here that as a representation,

$$T^{\sigma} \otimes F = \mathrm{Ind}_P^G \, (\sigma \otimes (F|_{MA})),$$

because this is valid at character level and $(*) \otimes F$ is an exact functor.

§ 2. Explicit calculations in the Lorentz group of 5-th order

In this section, we consider some examples of the decomposition of tensor product $E_\lambda \otimes F$, where E_λ is an infinite dimensional representation of L_5, and F is a finite dimensional one.

2.1. Let $G = L_5$ be the Lorentz group of 5-th order, i.e.,

$$L_5 = \{g \in SL(5,\mathbb{R}) \mid {}^t g J g = J,\ g_{55} \geqq 1\},$$

where $g = (g_{ij})_{1 \leqq i,j \leqq 5}$ and

$$J = \begin{pmatrix} 1_4 & 0 \\ 0 & -1 \end{pmatrix},$$

(we denote by 1_m the identity matrix of order m). This group has two conjugacy classes of Cartan subgroups. One of them is the class of compact Cartan subgroups and the other is the class of non-compact ones. The rank of L_5 is two and Weyl group is isomorphic to $\mathfrak{S}_2 \ltimes (\mathbb{Z}_2)^2$ (semi-direct product) where $\mathfrak{S}_2$ is the symmetric group of order 2 and $\mathbb{Z}_2$ is the multiplicative group $\{\pm 1\}$.

2.2. Let us explain the set of irreducible representations of L_5. All the irreducible admissible representations of L_5 are classified into the four classes of representations listed below (see [5],[6]).

(1) Finite dimensional representation $S_{(n_1,n_2)}$, where n_1 and n_2 are integers such that $0 \leqq n_1 \leqq n_2$. This is the representation of highest weight $\mu = (n_1, n_2)$.

(2) Principal series representation $T^{(\alpha;c)}$, where α is a non-negative integer and c is a complex number. If we consider α as the irreducible representation of $SO(1) \simeq M$ with highest weight α and c as the one-dimensional representation of $\mathbb{R} \simeq A$, then we have $T^{(\alpha;c)} = \mathrm{Ind}_P^G\, \alpha \otimes c$ (for notations M, A, P, see 1.2). The representation $T^{(\alpha;c)}$ is irreducible

if c is not a half integer or $c = \alpha + 1/2$.

(3) Discrete series representations $D^{+}_{(\alpha;p)}$ and $D^{-}_{(\alpha;p)}$, where α is a positive integer and p is an integer such that $0 < p \leqq \alpha$. These are the discrete series representations corresponding to $\lambda = (\alpha+1/2,\ p-1/2)$ (see 1.1). There are two series of irreducible square-integrable representations for L_5, and we denote by D^{+}_{*} one of the series corresponding to the positive chamber $C^{+} = \{(n_1,\ n_2) \mid 0 \leqq n_1 \leqq n_2\}$ and by D^{-}_{*} the other series corresponding to the chamber $C^{-} = \{(n_1,\ n_2) \mid 0 \leqq -n_1 \leqq n_2\}$.

(4) Representation $D^{1}_{(\alpha;p)}$, where α is a non-negative integer such that $0 \leqq p < \alpha$. This is the quotient representation $T^{(p;\alpha+1/2)}/S_{(p,\alpha-1)}$.

We remark here that (1)-(4) are disjoint classes of representations and all the representations of (2)-(4) are irreducible. For the class (1), the sufficient condition for irreducibility given there is also necessary. The representation $T^{(\alpha;c)}$ is contragredient to $T^{(\alpha;-c)}$, so we have $T^{(\alpha;c)} \simeq T^{(\alpha;-c)}$ in case that $T^{(\alpha;c)}$ is irreducible. Except for this case, the representations of (1)-(4) are mutually inequivalent.

2.3. Let us show examples.

(i) $D^{+}_{(1;1)} \otimes S_{(0,1)} = (D^{1}_{(1;0)};\ 2D^{+}_{(1;1)}) \oplus D^{+}_{(2;1)}$,

where $(D^{1}_{(1;0)};\ 2D^{+}_{(1;1)})$ is the representation whose composition factors are $D^1_{(1;0)}$ and two times $D^+_{(1;1)}$. On compact Cartan subgroups, this becomes

$$\text{(i)'} \quad \theta(D^+_{(1;1)} \otimes S_{(0,1)}) = \theta(D^+_{(1;1)}) - \theta(D^-_{(1;1)}) + \theta(D^+_{(2;1)}).$$

$$\text{(ii)} \quad D^-_{(1;1)} \otimes S_{(0,1)} = (D^1_{(1;0)};\ 2D^-_{(1;1)}) \oplus D^-_{(2;1)}.$$

In general, composition factors for $D^-_* \otimes F$ are the same as those for $D^+_* \otimes F$ except for the factors D^+_* and D^-_*. More precisely, we can get the composition factors of $D^-_* \otimes F$ replacing D^+_* and D^-_* in those of $D^+_* \otimes F$.

$$\text{(iii)} \quad D^+_{(1;1)} \otimes S_{(0,2)} = (D^1_{(1;0)};\ 3D^+_{(1;1)}) \oplus (D^1_{(2;0)};\ 2D^+_{(2;1)}) \oplus D^+_{(3;1)} \oplus T^{(1;3/2)}.$$

This shows that (discrete series representation) $\otimes F$ can contain an irreducible principal series representation which is induced from a minimal parabolic subgroup.

$$\text{(iv)} \quad T^{(1;0)} \otimes S_{(0,1)} = T^{(0;0)} \oplus T^{(1;0)} \oplus T^{(2;0)} \oplus 2T^{(1;1)}.$$

$$\text{(v)} \quad T^{(1;3/2)} \otimes S_{(0,1)} = T^{(1;3/2)} \oplus (S_{(1,1)};\ 2D^1_{(2;1)};\ D^+_{(2;2)};\ D^-_{(2;2)}) \oplus (S_{(0,0)};\ 2D^1_{(1;0)};\ D^+_{(1;1)};\ D^-_{(1;1)}).$$

This is refinement of the decomposition in Proposition 1.2.

In fact, we have

$$(v)' \quad \theta(T^{(1;3/2)} \otimes S_{(0,1)}) = \theta(T^{(1;3/2)}) + \theta(T^{(2;3/2)}) +$$

$$+\theta(T^{(0;3/2)}) + \theta(T^{(1;5/2)}) + \theta(T^{(1;1/2)}).$$

2.4. As we mentioned in § 0, in the case that G is the Lorentz group of n-th order, we can decompose the tensor product $E_\lambda \otimes F$ explicitly (at the character level). For detailed discussion, see [8].

Appendix. Tables of decompositions of $E_\lambda \otimes F$ for $SL(2,\mathbb{R})$

We summarize the decompositions of $E_\lambda \otimes F$ into two tables in case that $G = SL(2,\mathbb{R})$.

Let $SL(2,\mathbb{R}) = KAN$ be an Iwasawa decomposition where

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \; 0 \leqq \theta < 2\pi \right\}, \quad A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| \; t \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \middle| \; \xi \in \mathbb{R} \right\}.$$

If we put $M = Z_K(A)$ then $M = \{\mathrm{diag}(c,c) \mid c = \pm 1\}$, the center of G. Put $P = MAN$, a minimal parabolic subgroup of G. We list below the irreducible representations of $SL(2,\mathbb{R})$ (see [9], [10]).

(1) Principal series representation $T^{(\varepsilon,\alpha)} = \mathrm{Ind}_P^G \, \omega_{\varepsilon,\alpha}$

where $\omega_{\varepsilon,\alpha}$ is the one-dimensional representation of P such that ($\varepsilon = 0, 1$ and $\alpha \in \mathbb{C}$)

$$\omega_{\varepsilon,\alpha}\left(\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\right) = c^{\varepsilon}\, e^{\alpha t}.$$

(2) Discrete series representations, $D^{+}_{\varepsilon,2m+\varepsilon}$ and $D^{-}_{\varepsilon,2m+\varepsilon}$ ($m \geqq 1$, $\varepsilon = 0, 1$). These representations are subrepresentations of $T^{(\varepsilon,2m+\varepsilon)}$ and $D^{+}_{\varepsilon,2m+\varepsilon}$ is a holomorphic discrete series and $D^{-}_{\varepsilon,2m+\varepsilon}$ is an anti-holomorphic one.

(3) Limits of discrete series representations, $LD^{+}_{1,0}$ and $LD^{-}_{1,0}$. These representations give the direct decomposition of $T^{(1,0)}$:

$$T^{(1,0)} = LD^{+}_{1,0} \oplus LD^{-}_{1,0}$$

and $LD^{+}_{1,0}$ is holomorphic and $LD^{-}_{1,0}$ is anti-holomorphic.

(4) Finite dimensional representation F_m ($m \geqq 1$). This representation is the m-dimensional irreducible representation of $SL(2,\mathbb{R})$.

We put $\varphi(n) = (1 + (-1)^{n})/2$ for brevity.

<u>Table A.1.</u> (principal series representation)$\otimes$ F

Condition for ε,α,m	Decomposition of $T^{(\varepsilon,\alpha)} \otimes F_m$
$\varepsilon = 0$ $\alpha \notin \mathbb{Z}$ or $\alpha \geqq m$ and $\alpha \notin 2\mathbb{Z}+1$	$\sum_{j=0}^{m-1} \oplus\, T^{(\varphi(m+\varepsilon),\alpha-m+2j+1)}$

Condition for ε,α,m	Decomposition of $T^{(\varepsilon,\alpha)} \otimes F_m$
$\varepsilon = 1$ $\alpha \notin \mathbb{Z}$ or $\alpha \geqq m$ and $\alpha \in 2\mathbb{Z}+1$	$\sum_{j=0}^{m-1} \oplus\, T^{(\varphi(m+\varepsilon),\alpha-m+2j+1)}$
$\varepsilon = 0$ $0 \leqq \alpha \leqq m-1$ and $\alpha \in 2\mathbb{Z}$ $m \in 2\mathbb{Z}+1$ ——— $\varepsilon = 1$ $0 \leqq \alpha \leqq m-1$ and $\alpha \in 2\mathbb{Z}+1$ $m \in 2\mathbb{Z}$	$T^{(0,0)} \oplus \sum_{j=1}^{(m-\alpha-1)/2} \oplus\, 2T^{(0,2j)} \oplus$ $\oplus \sum_{j=(m-\alpha+1)/2}^{(m+\alpha-1)/2} \oplus\, T^{(0,2j)}$
$\varepsilon = 0$ $0 \leqq \alpha \leqq m-1$ and $\alpha \in 2\mathbb{Z}$ $m \in 2\mathbb{Z}$ ——— $\varepsilon = 1$ $0 \leqq \alpha \leqq m-1$ and $\alpha \in 2\mathbb{Z}+1$ $m \in 2\mathbb{Z}+1$	$\sum_{j=0}^{(m-\alpha)/2-1} \oplus\, 2T^{(1,1+2j)} \oplus$ $\oplus \sum_{j=(m-\alpha)/2}^{(m+\alpha)/2-1} \oplus\, T^{(1,1+2j)}$

Table A.2. (discrete series representation)$\otimes F$

We put $\delta = \pm$ (signature) in this table.

Condition for n, m	Decomposition of $(L)D^{\delta}_{\varphi(n),n} \otimes F_m$
$n \geqq m$	$\sum_{j=0}^{m-1} \oplus\, D^{\delta}_{\varphi(n+m+1),n-m+2j+1}$
$m > n \geqq 0$ $n+m \in 2\mathbb{Z}$	$\sum_{j=0}^{(m-n)/2-1} \oplus\, (2D^{\delta}_{0,1+2j};\ F_{1+2j}) \oplus$ $\oplus \sum_{j=(m-n)/2}^{(n+m)/2-1} \oplus\, D^{\delta}_{0,1+2j}$

Condition for n,m	Decomposition of $(L)D^{\delta}_{\varphi(n),n} \otimes F_m$
$m > n \geqq 0$ $n+m \in 2\mathbb{Z}$	$LD^{\delta}_{1,0} \oplus \sum_{j=1}^{(m-n-1)/2} \oplus (2D^{\delta}_{1,2j};\ F_{2j}) \oplus$ $\oplus \sum_{j=(m-n+1)/2}^{(n+m-1)/2} \oplus D^{\delta}_{1,2j}$

References

[1] I.N.Bernstein and S.I.Gel'fand, Tensor products of finite and infinite dimensional representation of semisimple Lie algebras, Composit. Math., 41(1980), 245-285.

[2] I.N Bernstein, I.M.Gel'fand and S.I.Gel'fand, Structure of representations generated by vectors of highest weight, Funct. Anal. and Appl., 5(1971), 1-8.

[3] T.J.Enright, Lectures on representations of complex semisimple Lie groups, Tata Inst. Fund. Research, Springer-Verlag, 1981.

[4] Harish-Chandra, Discrete series for semisimple Lie groups II, Acta Math., 116(1966), 1-111.

[5] T.Hirai, On irreducible representations of the Lorentz group of n-th order, Proc. Japan Acad., 38(1962), 258-262.

[6] T.Hirai, Relation between certain standard representations of $\underline{g}$ and non-unitary principal series representations of G. Case of $G = SO_0(n-1,1)$, $\underline{g} = so(n-1,1)$, RIMS kôkyûroku, 300(1977), 54-70.

[7] A.W.Knapp and G.Zuckerman, Classification of irreducible tempered representations of semisimple groups I,II, Ann. Math., 116(1982), 389-455, 457-501.

[8] K.Nishiyama, Deconpositions of tensor products of infinite and finite dimensional representations of semi-simple groups, preprint.

[9] S.Okamoto, 等質空間上の解析学, 紀伊國屋, 1980.

[10] M.Sugiura, Unitary representations and harmonic analysis, Kodansha, 1975.

[11] G.Zuckerman, Tensor product of finite and infinite dimensional representations of semisimple Lie groups, Ann. Math., 106(1977), 295-308.