

BLOW-UP RESULTS FOR A REACTION-DIFFUSION SYSTEM

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ABSTRACT. We consider the initial value problem for the reaction-diffusion system with the nonlinear terms $|x|^{\sigma_j} u^{p_j} v^{q_j}$. In this system, the exponents p_1 and q_2 play a crucial role to determine the behavior of the solutions. Using an ODE method, we prove the Fujita-type nonexistence results.

We consider the Cauchy problem for the reaction-diffusion system:

$$\begin{aligned} (1) \quad & u_t - \Delta u = |x|^{\sigma_1} u^{p_1} v^{q_1}, & x \in \mathbf{R}^N, \quad t > 0, \\ (2) \quad & v_t - \Delta v = |x|^{\sigma_2} u^{p_2} v^{q_2}, & x \in \mathbf{R}^N, \quad t > 0, \\ & u(x, 0) = u_0(x) \geq 0, \neq 0, & x \in \mathbf{R}^N, \\ & v(x, 0) = v_0(x) \geq 0, \neq 0, & x \in \mathbf{R}^N, \end{aligned}$$

where $p_j, q_j \geq 0$, $\sigma_j > \max(-2, -N)$ ($j = 1, 2$), and $p_1, q_2 \neq 1$.

Our aim is to show the conditions for the nonexistence of global solutions of the system (1) and (2). The conditions are about the relation between the exponents p_j, q_j, σ_j , and the initial data.

There are some papers on the Cauchy problem for semilinear reaction-diffusion systems. In [2], Escobedo and Herrero proved the existence and nonexistence of global solutions, so-called the Fujita-type result, for $\sigma_1 = \sigma_2 = p_1 = q_2 = 0$, $p_2, q_1 \geq 1$, $p_2 q_1 > 1$. As an extension of [2], Mochizuki and Huang [4] showed the Fujita-type result for $p_1 = q_2 = 0$, $0 \leq \sigma_1 < N(p_2 - 1)$, $0 \leq \sigma_2 < N(q_1 - 1)$, $p_2, q_1 \geq 1$, $p_2 q_1 > 1$. Both of the results show that the interaction between the unknown functions in the nonlinear terms determines the behavior of solutions of the system.

In [3], Escobedo and Levine showed an interesting result for $\sigma_1 = \sigma_2 = 0$, $p_1, p_2, q_1, q_2 \geq 0$. Under the assumption that $p_2 + q_2 \geq$

$p_1 + q_1 > 0$, they showed that if $p_1 > 1$, the solutions of the system behave like a solution of the single equation $u_t - \Delta u = u^{p_1+q_1}$.

In fact, the same result as [3] holds in our problem, that is, if $p_1 > 1$, the solutions of the system behave like a solution of the single equation $u_t - \Delta u = |x|^{\sigma_1} u^{p_1+q_1}$ under the assumption that $(p_2 + q_2 - 1)/(\sigma_2 + 2) \geq (p_1 + q_1 - 1)/(\sigma_1 + 2)$.

The iteration method of [3] is often used to show blow up for reaction-diffusion systems. However, the method does not seem applicable for our problem because the nonlinear terms have the variable coefficients $|x|^{\sigma_j}$. We improve the argument in [4] and apply it to our problem. The argument in [4] is to transform the system of PDEs into the ordinary differential inequalities. In our problem, multiplying the equation by negative power of unknown function makes the transformation possible.

For simplicity, let

$$\begin{cases} \alpha = \frac{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2)}{2\{p_2q_1 - (1 - p_1)(1 - q_2)\}}, \\ \beta = \frac{p_2(\sigma_1 + 2) + (1 - p_1)(\sigma_2 + 2)}{2\{p_2q_1 - (1 - p_1)(1 - q_2)\}}, \\ \delta_1 = \frac{q_1\sigma_2 + (1 - q_2)\sigma_1}{p_2q_1 - (1 - p_1)(1 - q_2)}, \\ \delta_2 = \frac{p_2\sigma_1 + (1 - p_1)\sigma_2}{p_2q_1 - (1 - p_1)(1 - q_2)}. \end{cases}$$

For $a \in \mathbf{R}$, we define the function spaces:

$$I^a = \{w \in C(\mathbf{R}^N); w(x) \geq 0, \limsup_{|x| \rightarrow \infty} |x|^a w(x) < \infty\}.$$

We assume that the initial data $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$.

Theorem 1. *Assume that*

$$(3) \quad \frac{p_1 + q_1 - 1}{\sigma_1 + 2} \leq \frac{p_2 + q_2 - 1}{\sigma_2 + 2},$$

and let $p_1 < 1$, $q_2 \neq 1$.

- (i) *If $\max(\alpha, \beta) \geq N/2$, then no nontrivial global solutions exist.*
- (ii) *If $0 < \max(\alpha, \beta) < N/2$, then no global solutions exist for large initial data.*

Theorem 2. Assume (3), and let $p_1 > 1$, $q_2 \neq 1$.

- (i) If $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$, then no nontrivial global solutions exist.
- (ii) If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then no global solutions exist for large initial data.

On the other hand, authors in [1] show the conditions for the existence of global solutions of the system as follows.

Theorem 1'. Assume (3), and let $p_1 < 1$, $q_2 \neq 1$, $\sigma_j \geq 0$ ($j = 1, 2$).

- (i) If $0 < \max(\alpha, \beta) < N/2$, then global solutions exist for small initial data.
- (ii) If $\max(\alpha, \beta) < 0$, then every solution is global.

Theorem 2'. Assume (3), and let $p_1 > 1$, $q_2 \neq 1$, $\sigma_j \geq 0$ ($j = 1, 2$).

If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then global solutions exist for small initial data.

Comparing Theorems 1 and 2 with the results in [1], we can see that our conditions are optimal.

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