

On subriemannian contact manifolds

by

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Subriemannian geometry has recently attracted a great deal of attention by new phenomena never arising in riemannian geometry but peculiar to subriemannian geometry. A subriemannian manifold (M, D, g) is a differential manifold M endowed with a subbundle D of the tangent bundle TM and a riemannian metric g on D . (A *riemannian metric* on a bundle D is a smooth section $p \rightarrow g_p$ of the bundle $D^* \otimes D^*$ such that for each $p \in M$ the bilinear form $D(p) \times D(p) \ni (v, w) \rightarrow g_p(v, w) \in \mathbf{R}$ is symmetric and strictly positive definite. It is clear that one can always construct a riemannian metric on any subbundle D of TM by just taking a riemannian metric on TM and restricting it to D . Conversely, every riemannian metric on a smooth distribution D on M arises in this way, as can be shown by using partitions of unity.) We say that a subriemannian manifold (M, D, g) is isomorphic to (M', D', g') if there exists a diffeomorphism $\varphi : M \rightarrow M'$ such that $\varphi_*D = D'$, $\varphi^*g' = g$. Contrary to riemannian geometry, it is not a priori clear whether the automorphism group $\text{Aut}(M, D, g)$ of a subriemannian manifold (M, D, g) makes a finite dimensional Lie group.

A local vector field X on M is called an infinitesimal automorphism of (M, D, g) if $L_X D \subset D$ and $L_X g = 0$. Let \mathcal{L} be the sheaf of germs of infinitesimal automorphisms of (M, D, g) and \mathcal{L}_a the stalk of \mathcal{L} at $a \in M$.

If the evaluation map $\mathcal{L}_a \ni [X]_a \mapsto X_a \in T_a M$ is surjective for all $a \in M$ (where we denote by $[X]_a$ the germ of X at a and by X_a the value of X at a), we say that \mathcal{L} is transitive or that (M, D, g) is homogeneous. Note that if M is connected and if \mathcal{L} is transitive then $\text{Aut}(M, D, g)$ is transitive and all stalks of \mathcal{L} are isomorphic to each other.

Simple but non-trivial and interesting class of subriemannian manifolds may be that of the subriemannian contact manifolds: A subriemannian manifold (M, D, g) is called a subriemannian contact manifold if D is a contact structure, that is, of codimension 1 and non-degenerate.

In this talk we study the structure of the Lie algebra \mathcal{L}_a for a point a of a homogeneous subriemannian contact manifold (M, D, g) of dimension $2n + 1$ through the viewpoint of nilpotent geometry ([3], [4]).

We first introduce the contact filtration $\{\mathcal{L}_a^p\}_{p \in \mathbf{Z}}$ of \mathcal{L}_a . Then, passing to the projective limit: $L = \varprojlim_k \mathcal{L}_a / \mathcal{L}_a^k$, $L^p = \varprojlim_k \mathcal{L}_a^p / \mathcal{L}_a^k$, we obtain a Lie algebra L and its filtration $\{L^p\}_{p \in \mathbf{Z}}$. The filtered Lie algebra $(L, \{L^p\})$ is called the formal algebra of \mathcal{L} at a and is a transitive filtered Lie algebra (TFLA) of depth 2 in the sense of Morimoto [3]. In fact, the formal algebras L arising from the homogeneous subriemannian contact manifolds form a special class of TFLA's and are called the subriemannian contact TFLA's.

We study structures of subriemannian contact TFLA's, following the general theory for TFLA's developed in [3]. We first show that any subriemannian contact TFLA L is finite dimensional and of dimension not greater than $(n + 1)^2$. Moreover, if L is maximal, that is, L attains the maximal dimension $(n + 1)^2$, then the associated graded Lie algebra grL is isomorphic to a unique transitive graded Lie algebra $\mathfrak{k} = \bigoplus_p \mathfrak{k}_p$.

Next we shall determine concretely the structures of maximal subriemannian contact TFLA's K . In order to do this, we first examine the cohomology group associated with the graded Lie algebra \mathfrak{k} . Then, on the basis of these results, we obtain our main theorem. In particular, we shall show: *If K is a maximal subriemannian contact TFLA, then K is isomorphic to the graded Lie algebra \mathfrak{k} , $\mathfrak{u}(n + 1)$, or $\mathfrak{u}(n, 1)$.*

It is shown by Morimoto [5] that there exists a canonical Cartan connection associated with a subriemannian manifold satisfying certain regularity

conditions. Using this Cartan connection, we see, in particular, that the stalk \mathcal{L}_a is isomorphic to its formal algebra L . Thus the above results for the formal algebras L also hold for \mathcal{L}_a .

References

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