

Canonical filtrations and stability of direct images by Frobenius morphisms

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There is an important concept of stability for vector bundles or torsion free sheaves on algebraic varieties. In this talk, we introduce a problem about stability of direct images by Frobenius morphisms. This is a joint work with H. Sumihiro.

Let k be an algebraically closed field of characteristic $p > 0$, X a nonsingular projective variety over k of dimension n , $F = F_X$ the absolute Frobenius morphism of X and H be an ample divisor on X . Then one can define the *slope* of a torsion free sheaf \mathcal{E} on X with respect to H by

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E})H^{n-1}}{\mathrm{rk}(\mathcal{E})},$$

where $\mathrm{rk}(\mathcal{E})$ is the rank of \mathcal{E} . Then a torsion free sheaf \mathcal{E} on X is called *semistable* (respectively, *stable*) with respect to H if for all nonzero torsion free subsheaf \mathcal{F} of \mathcal{E} , $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (respectively, $\mu(\mathcal{F}) < \mu(\mathcal{E})$).

H. Lange and C. Pauly proved the following theorem:

Theorem 1 (H. Lange, C. Pauly [4]). *Let X be a nonsingular projective curve over k of genus $g(X) \geq 2$ and \mathcal{L} be a line bundle on X . Then $F_*\mathcal{L}$ is stable.*

Then we can consider following natural question:

Problem 2. *Let X be a nonsingular projective variety of general type over k of dimension $n \geq 2$, \mathcal{L} be a line bundle on X , and H be an ample line bundle on X . Is $F_*\mathcal{L}$ semistable with respect to H ?*

We compute the first Chern classes of direct images of vector bundles by Frobenius morphisms. K_X denotes the canonical divisor of X .

Theorem 3. *Let \mathcal{E} be a vector bundle on X of rank r . Then*

$$c_1(F_*\mathcal{E}) \underset{\text{num}}{\equiv} \frac{p^n - p^{n-1}}{2} r K_X + p^{n-1} c_1(\mathcal{E}),$$

where $\underset{\text{num}}{\equiv}$ denotes numerical equivalence.

We introduce a useful filtration of $F^*F_*\mathcal{O}_X$. There is a natural surjection $F^*F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$. Let I be the kernel of this. Since $F^*F_*\mathcal{O}_X$ has the structure of \mathcal{O}_X -algebra, we get a filtration

$$F^*F_*\mathcal{O}_X \supset I \supset I^2 \supset I^3 \supset \dots$$

of $F^*F_*\mathcal{O}_X$. Here, we call this filtration *canonical filtration* of $F^*F_*\mathcal{O}_X$.

Theorem 4. *Let X be a nonsingular projective surface over k and I^\bullet be the canonical filtration of $F^*F_*\mathcal{O}_X$. Then*

$$F^*F_*\mathcal{O}_X \supset I \supset I^2 \supset \dots \supset I^{2p-2} \supset I^{2p-1} = (0)$$

and

$$I^i/I^{i+1} = \begin{cases} S^i(\Omega_X^1) & (0 \leq i \leq p-1), \\ K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega_X^1) & (p \leq i \leq 2p-2). \end{cases}$$

Using Theorem 4, we can prove following theorem.

Theorem 5. *Let X be a nonsingular projective surface over k and H be an ample line bundle on X . Assume that K_X is nef and big and Ω_X^1 is semistable with respect to H . Then $F_*\mathcal{L}$ is semistable with respect to H for any line bundle \mathcal{L} on X .*

Theorem 6. *Let X be a nonsingular projective surface over k and H be an ample line bundle on X . Assume that $K_X \underset{\text{num}}{\equiv} 0$ and Ω_X^1 is semistable with respect to H . Then $F_*\mathcal{L}$ is semistable with respect to H for any line bundle \mathcal{L} on X .*

Moreover, using this filtration, we can give another simple proof of The Lange and Pauly's Theorem.

Furthermore, using Theorem 5, we can prove a result of geography of surfaces of general type.

References

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