On the blow-up at space infinity for solutions to the quasi-linear parabolic equations *

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1 Introduction

In this talk, the speaker would like to consider the blow-up problem for the quasilinear parabolic equations

$$u_t = \triangle u^m + u^p \qquad x \in \mathbb{R}^N,\ t > 0 \tag{1}$$

where m and p are physical constant such that $m > 1$ and $p > 1$ with initial value $u_0(x)$. The equation (1) is also called porous medium equation. It arises e.g., in the study of thermal diffusion phenomena with heat source. In this case, $u(x,t)$ repesents a temperature at the point x and at the time t.

Under the suitable condition on the intial data, the intial value problem (1) has a unique solution at least localy in time. Here the meaning of *solution* is understood in some weak sence.

The solution of (1) may not exist globally in time; it may blow up in finite time (the concrete definition of "blow up" is defined later). We are interested in the blow-up at space infinity and nonblow-up for the solution of (1).

The blow-up problem has been studied for a long time since Fujita's classical paper [2] at 1966 was published. He considered the initial value problem for the semilinear parabolic equations

$$u_t = \triangle u + u^p, \quad x \in \mathbb{R}^N,\ t > 0 \tag{2}$$

where $p > 1$ with initial value $u(x,0) = u_0(x)$. If u_0 is bounded and continuous in $\mathbb{R}^N$, there exsists a unique solution of the (2) at least locally in time

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and it can be extended as time increase as far as $u(\cdot,t)$ belongs to L^∞. He discovered the exponent

$$p_f = 1 + \frac{2}{N}$$

, in which the situation extremely changes: When $p < p_f$, none of the solutions of (2) exsist globally in time except for the trivial one $u \equiv 0$ no matter how the initial value u_0 is small. On the other hands when $p > p_f$, the soution of (2) can be global in time if the initial value u_0 is small enough. In the first case the solution is said to *blow up* in finite time. After that, it was also shown that the solution of (2) blows up in finite time in the case of $p = p_f$ for $N = 1, 2$ by [7]. Thus the number p_f is called "critial exponent" or "Fujita exponent". Similar results were shown for the quasilinear parabolic equation (1) by [3]. Let us put

$$p^* \equiv p^*(m, N) = m + \frac{2}{N}.$$

When $p < p^*$, none of the solutions of (1) exisit globally in time except for the trivial one $u \equiv 0$ no matter how the initial value u_0 is small. On the other hands, when $p \geq p^*$, the solution of (1) can be global in time if the initial value u_0 is small enough.

The various problems concerning blow-up for (1) and (2) have been studied. The one of these problems is determing the locations of *blow-up points* defined below. *Blow-up* and *blow-up point* are defined for the solution of (1) as follows: For a given u_0, m and p, let $T^* = T^*(u_0, m, p)$ be a maximal existence time of the solution of (1). If $T^* = \infty$, we call the solution exists globally in time. When $T^* < \infty$, we say that the solution blows up in finite time and call T^* *blow-up time* of the solution. By this definition, it follows that

$$\lim_{t \uparrow T^*} \sup_{x \in \mathbb{R}^N} |u(x,t)| = \infty.$$

Because if it does not hold, the solution will be uniquely prolonged. A point $x_{BU} \in \mathbb{R}^N$ is called *blow-up point* (at the time T^*) of the solution provided there exists a sequence $\{(x_n, t_n)\}$ such that

$$t_n \uparrow T^*, \quad x_n \to x_{BU} \quad \text{and} \quad u(x_n, t_n) \to \infty \quad \text{as} \quad n \to \infty.$$

We shall denote by S the set of all blow-up points of u and call it *blow-up set* of u. If there exists a sequence $\{(x_n, t_n)\}$ satisfying

$$t_n \uparrow T^*, \quad |x_n| \to \infty \quad \text{and} \quad u(x_n, t_n) \to \infty \quad \text{as} \quad n \to \infty,$$

we say that the solution blows up at space infinity (at the time T^*).

There is a huge literature on the locations of blow-up points since earlyer work by Weissler [9] and A. Friedman and B. Mcleod [1]. Here we shall introduce the results about this topic by Mochizuki and Suzuki [8]. They treated the Cauchy problems (1) as examples and classified the blow-up sets in terms of the relations between m and p: Assume that $u_0 \geq 0$, $u_0 \not\equiv 0$ and u_0 is bounded continuous in $\mathbb{R}^N$. Suppose that u_0 has a compact support in $\mathbb{R}^N$ or decays at space infinity. Note that if $p \leq m$, then all nontrivial solution of the (1) blow up in finite time.

i) *Let $m > p$ and u be a blow-up solution of* (1). *Then u blows up at any points in $\mathbb{R}^N$, in other words,*

$$S = \mathbb{R}^N.$$

Moreover u blows up uniformly in each compact set K of $\mathbb{R}^N$:

$$\lim_{t \uparrow T^*} \inf_{x \in K} u(x,t) = \infty.$$

ii) *Let $p = m$ and u be a blow-up solution of* (1). *Then S includes some ball.*

iii) *Let $p > m$ and u be a blow-up solution of* (1). *Then S is included in a domain depending only on the shape of the initial data $u_0(x)$. Especially, if u_0 is radially symmetric and decreasing with respect to origin, then so called, single point blow-up occurs i.e.,* $S = \{0\}$.

These results represent quite defferences between semilinear case and quasilinear case. But it is always imposed that u_0 is decaying at space infinity.

We would like to consider the initial value problems (1) for more gerneral initial datas. Let us consider a continuous function u_0 in $\mathbb{R}^N$ which satisfies

$$0 \leq u_0(x) \leq M \quad \text{for all } x \in \mathbb{R}^N \tag{A1}$$

$$\lim_{|x| \to \infty} u_0(x) = M \tag{A2}$$

for some $M > 0$.

We compare (1) with the associating initial value problem for the ordinary differential equation;

$$\begin{cases} v' = v^p, \\ v(0) = M, \end{cases} \tag{3}$$

The initial value problem (3) is immediately solved,

$$v(t) = \frac{\alpha^\alpha}{(T_v - t)^\alpha}$$

where

$$\alpha = \frac{1}{p-1} \quad \text{and} \quad T_v = \frac{\alpha}{M^{p-1}} \quad \text{: maximal existence time of } v.$$

The solution of (3) is considered to be the solution of (1) which is spatially constant for all $\mathbb{R}^N$ and necessarily blows up at the time T_v. We are now in the position to state our main results. We restrict ourselves in the case of $1 \leq m < p$.

Theorem 1. *Let u be a solution of* (1) *with* (A1) *and* (A2)*. Then u blows up at $T^* = T_v$ and satisfies*

$$\lim_{|x|\to\infty} u(x,t) = v(t) \tag{4}$$

and the convergence is uniform in every compact subset of $[0, T_v)$*.*

In fact, Theorem 1 ensures that the solution of (1) with (A1) and (A2) at least blows up at space infinity and describes its asymptotic behaviour as $|x| \to \infty$. On the other hands, we can obtain the fact that the solution of (1) does not blow up at any points in $\mathbb{R}^N$ under the assumptions (A1) and (A2):

Theorem 2. *Let u be a solution of* (1) *with* (A1) *and* (A2) *and assume that $u_0 \not\equiv M$. Then u has no blow up points in $\mathbb{R}^N$. In other words, u blows up only at space infinity.*

There are few works on the solution which blows up at space infinity. We mention that this problem is discussed by Y. Giga and N. Umeda in [4] and [5] for the semilinear case. They proved in [4] above Theorems in the case of $m = 1$. But their proofs heavily depend on the fundamental solution for the heat equation. However one can not use same method for the case $m > 1$ since there are no fundamental solutions to the quasilinear equations.

Let us describe a brief sketch of our method. To show Theorem 1, we adopt comparison arguments. We can easily see the fact that the solution of (1) is estimate from above by the solution of (3). We shall focus atention on the construction of the subfunction which converges to the the solution of (3) uniformly as $|x| \to \infty$. From the assumption (A1), u_0 takes very close value to M at far away from origin. So we consider the zero-Dirichlet boundary value problems on the balls centered at the points far away from origin and estimate these solutions from below making use of, so called, "Kaplan's method" which is using the eigenfunction of $-\triangle$ for Dirichlet problem as suitable test functions (c.f. [6]). Here it is essentially used that $p \geq m$. We adopt comparison arguments also to prove Theorem 2. We construct a supersolution for the investigated Cauchy problem which has no blow-up points in $\mathbb{R}^N$ and use the methods having been established in [1], [8], and so on.

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