A REMARK ON LAPLACE EIGENFUNCTIONS ON M_A^3

YOSIHISA MIYANISI

Department of Mathematics, Faculty of Science
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku, Tokyo, 152-8551, Japan

ABSTRACT. M_A^3 is a compact 3-dimensional solvable manifolds without boundary. In this talk, we shall review the specturm and the eigenfunctions of the corresponding Laplacian (See [Bo-Du-Ve]). First we would like to see that the geodesic flow on M_A^3 is completely integrable, but the level spacing distribution for the spectrum of M_A^3 is not Poisson. Hence Berry-Tabor conjecture does not hold in this case. Finally we introduce the equi-distributed eigenfunctions and the concentrated eigenfunctions on M_A^3.

1. INTRODUCTION AND RESULTS.

It has been known since the nineteenth century that in dimension two there is a close relationship between geometry and topology. Namely each compact orientable manifolds admits a metric of constant curvature: positive if it is a topologicalsphere, zero if it is a torus and negative if it has genus more than 1.

In dimension three the situation is much more sophisticated. Thurston [Th] put forward the famous Geometriation conjecture: any compact orientable 3-manifolds can be cut by disjoint embeddedd 2-spheres and tori into pieces, which after glueing 3-balls to all boundary spheres, admit one of 8 special geometric structures. These special 3-dimensional geometries are the standard Euclidean $\mathbf{R}^3$, spherical S^3 and Hyperbolic $\mathbf{H}^3$ geometries, the product geometries $S^2 \times \mathbf{R}$ and $H^2 \times \mathbf{R}$ and three geometries related to the Lie groups $SL_2(\mathbf{R})$, Nil and Sol. In this talk we restrict our concern to the main class of Sol-manifolds M_A^3.

On the other hand, in the asymptotic theory of high-frequency eigefunctions u_k of Laplace operator the most attention was paid to the quasiclassic eigenfunctions. They are associated with the simplest invariant sets of the geodesic flow, namely stable closed orbits and invariant tori.

The following examples are typical:

Example 1. *If the geodesic flow on the cosphere bundle S^*M is ergodic, there exists a subsequence satisfying $|u_{k_j}(x)|^2 d\,vol_M \to \dfrac{d\,vol_M}{vol_M}$ as $j \to \infty$.* (See [Co],[Sc],[Ze]).

Example 2. *Let τ be a closed geodesic curve on the standard sphere S^2, there exists a subsequence satisfying $|u_{k_j}(x)|^2 d\,vol_M \to \delta_\tau$ as $j \to \infty$, where δ_τ denotes a measure distributed uniformly along τ.*

Date: January 25, 2006.
Key words and phrases. Sol-manifolds M_A^3.
The author would like to thank Professor A.Inoue for giving valuable comments.

Our main purpose is to investigate eigenfunctions on M_A^3.

For paritcular eigenfunctions (Bolsinov - Dullin - Veselov's eigenfunctions (See §3)), we see the equi-distributed eigenfunctions and the concentrated eigenfunctions on M_A^3.

Theorem 1 (Quantum ergodicity for Bolsinov - Dullin - Veselov's eigenfunctions)**.** *For fixed $\gamma \in L^*$, eigenfunctions ($-\triangle\psi_{\gamma,k} = E_{\gamma,k}\psi_{\gamma,k}$) are quantum-ergodic on M_A^3. (i.e. $|\psi_{\gamma,k_j}|^2 dvol_{M_A^3} \to \frac{dvol_{M_A^3}}{vol_{M_A^3}}$ as $j \to \infty$).*

Theorem 2 (Concentrated eigenfunctions)**.** *Let $\psi_{\gamma,k}$ be Bolsinov - Dullin - Veselov's eigenfunctions. Then there exists a subsequence γ_j and k_j such that $|\psi_{\gamma_j,k_j}|^2 dvol_{M_A^3} \to \delta_{T^2}$ as $j \to \infty$, where δ_{T^2} denotes a measure distributed uniformly along T^2.*

But, due to the high degeneracy of spectrum, there are infinite complete orthogonal bases of eigenfunctions. For all orthogonal bases, we have the following result.

Theorem 3. *Let $\{u_k\}$ be an arbitary orthogonal base of the Laplace eigenfunctions on M_A^3. Then there exists a subsequence such that $|u_{k_j}(x)|^2 d\,vol_M \to d\nu$ and the absolutely continuous part of ν is not 0.*

The structure of this note is following.

In §2, we shall review M_A^3 and some basic notations.

In §3, we define the Bolsinov - Dullin - Veselov's eigenfunctions and introduce the related results.

2. Sol-manifolds M_A^3 and the geodesic flow.

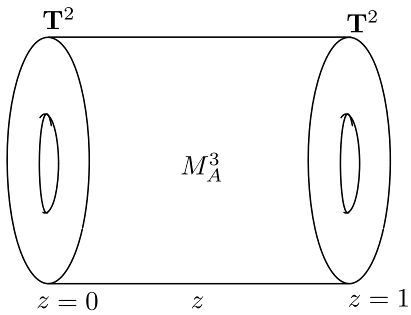
Sol-manifolds M_A^3 are T^2 torus bundles over a circle S^1 with hyperbolic glueing map with positive eigenvalues.

$$\mathrm{T_A} : \mathbf{T}^2 \times \mathbf{S}^1 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrowtail \begin{pmatrix} A\begin{pmatrix} x \\ y \end{pmatrix} \\ z+1 \end{pmatrix}$$

where $A \in SL^2(\mathbf{Z})$ is an integer hyperbolic matrix, which defines a hyperbolic automorphism of the 2-torus.

We define $M_A^3 = \mathbf{T}^2 \times \mathbf{S}^1 / \langle T_A \rangle$.

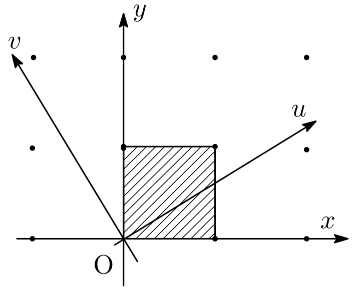
Figure 1.



Together with (x, y, z) we shall use another coordinate system (u, v, z) on M_A^3, where (u, v) are linear coordinates on the fibers related to a positively oriented eigenbasis e_u, e_v of A.

(i.e. $(x, y, z) \rightarrowtail (u, v, z)$, e_u and e_v are eigenbasis of A , see fig.2.)

Figure 2.



$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The corresponding metric on M_A^3 is defined by $ds^2 = e^{-2\mu z}Edu^2 + 2Fdu\,dv + e^{2\mu z}Gdv^2 + dz^2$ where λ is the largest eigenvalue of A and $\mu = \log\lambda$. E, F and G are constant. Thus the Hamiltonian of the geodesic flow on M_A^3 in (u, v, w)-coordinates can be written as

$$H = \frac{1}{2}(Ee^{2\mu z}p_u^2 + 2Fp_up_v + e^{-2\mu z}Gp_v^2 + p_z^2),$$

and furthermore,

$$\begin{cases} Q = p_up_v \\ F_1 = \sqrt{Q}e^{-1/Q^2}\cos\dfrac{\log(\sqrt{E/G}|p_u/p_v|)}{2\log\lambda} \\ F_2 = \sqrt{Q}e^{-1/Q^2}\sin\dfrac{\log(\sqrt{E/G}|p_u/p_v|)}{2\log\lambda} \end{cases}$$

are three global integrals in C^∞ category. So the geodesic flow on M_A^3 is integrable in C^∞ category.

Remark. The geodesic flow on M_A^3 is not integrable in analytic category (See [Ta]).

3. Spectrum and eigenfunctions on M_A^3.

Let us discuss the quantum problem on the Sol-Manifolds M_A^3:

$$-\triangle\psi = E\psi,$$

where $\triangle$ is the Laplace operator on M_A^3. In coordinates (u, v, z) the Laplace operator has the following explicit form:

$$\triangle = Ee^{2\mu z}\frac{\partial^2}{\partial u^2} + 2F\frac{\partial^2}{\partial u\partial v} + e^{-2\mu z}G\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial z^2}.$$

First we consider eigenfunctions on the covering space $\tilde{M_A^3} = T^2 \times \mathbf{R}$. Because the coefficents of $\triangle$ depends only on z, it is quite natural to separate variables and look for the eigenfunctions of $\triangle$ of the form

$$\Psi_{\gamma,k}(u, v, z) = e^{2\pi i(\gamma,w)}f_k(z) \text{ where } \gamma \in L^* \text{ (dual lattice of } T^2).$$

By substituing into the Shrödinger equation, we have

$$-\frac{d^2}{dz^2}f_k(z) + |\nu(\gamma)|\cosh(z + \alpha(\gamma))f_k(z) = E_kf_k(z) \quad \text{(modified Mathieu equation)}$$

where $\nu(\gamma)$ and $\alpha(\gamma)$ depend only on γ.

One can try to construct the genuine eigenfunctions of $\triangle$ on M_A^3 by averaging these eigenfunctions.

Definition (Bolsinov - Dullin - Veselov's eigenfunction).

$$\begin{cases} \tilde{\psi}_{\gamma,k} = \sum_{n\in\mathbf{Z}} \Psi_{(A^*)^n\gamma,k}(u,v,z+n) \\ \text{and} \\ \psi_{\gamma,k} = \tilde{\psi}_{\gamma,k}/\|\tilde{\psi}_{\gamma,k}\|_{L^2(M_A^3)}. \end{cases}$$

Because of the fast decay of $f_k(z)$ they are well defined on M_A^3. Some basic properties of $\psi_{\gamma,k}$ are known.

Theorem 4 (Bolsinov - Dullin - Veselov(preprint)). *The functions $\psi_{\gamma,k}$, $\psi_{0,k}$ form a complete orthogonal basis in $L^2(M_A^3)$.*

Next result is particular interesting.

Theorem 5 (Bolsinov - Dullin - Veselov(preprint)). *The level spacing distribution for the spectrum of M_A^3 is not Poisson.*

Note that according to the Berry-Tabor conjecture integrable systems should have Poisson distribute level spacing (See [Be-Ta]). This is not the case for Sol-manifolds.

Remark. Theorem 5 is not sensitive to change the metric $ds^2 = e^{-2\mu z}Edu^2 + 2Fdu\,dv + e^{2\mu z}Gdv^2 + dz^2$.

4. Concluding remarks

We reviewed that spectral stastics on M_A^3 provides a counterexample to the Berry-Tabor conjecture, but it cannnot be taken as an indicator of chaos. Some simple observations (Theorem1, Theorem 3) showed that the subset of eigenfunctions are asymptotically 'uniformly distributed' on the manifolds. Hence the subset of eigenfunctions are quantum ergodic. Thus M_A^3 is analogous to T^3 in these features (See [Bo], [Ja]). But theorem2 means that the subset of eigenfunctions are concentrated on T^2. M_A^3 and S^2 are analogous in this feature (See Example2).

So we conclude that M_A^3 has the middle property between two extremes.

References

[Be-Ta] M.V.Berry and M.Tabor *Level clustering in the regular spectrum*, Proc.Roy.Soc.A **356**, (1977), 375-394.

[Bo-Du-Ve] A.V.Bolsinov, H.R.Dullin, and A.P.Veselov *Spectra of Sol-manifolds: Arithmetic and quantum monodromy*, math-ph/0503046.

[Co] Colin de Verdiere, *Ergodicite et functions propres du Laplacien*, Commun.Math.Phys. **102**, (1985), 497-502.

[Bo] J.Bourgain *Eigenfunction bounds for the Laplacian on the n-torus*, Intern. Math. Res. Notices **3** (1993), 61-66.

[Ja] D.Jakobson *Quantum limits on flat tori*, Annal.Math. **145**, (1997), 235-266.

[Mi] Y.Miyanisi, *A note on Laplace eigenfunctions on M_A^3*, in preparation.

[Sc] A.I.Schnirelman, *Ergodic properties of eigenfunctions*, Usp.Math.Nauk **29**, (1974), 181-182.

[Ta] I.A.Taimanov *Topological obstructions to integrability of geodesic flows on non-simply-connected manifolds*, Math.USSR Izv. **30**, (1988), 403-409.

[Th] W.P.Thurston *Hyperbolic geometry and 3-manifolds*, London Math.Soc.Lecture note Ser.,48, Cambridge Univ. Press, (1982).

[Ze] S.Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke.Math.J. **55**, (1987), 919-941.

E-mail address: miyanisi@math.titech.ac.jp