

Large time behavior of solutions to a semilinear hyperbolic system with relaxation

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1 Introduction

We consider a nonlinear relaxation system of the form:

$$\begin{cases} u_t + v_x = 0, \\ v_t + u_x = f(u) - v. \end{cases} \quad (1.1)$$

The system (1.1) describes many physical phenomena such as nonequilibrium gas dynamics, magnetohydrodynamics, viscoelasticity, flood flow with friction, etc.

If we eliminate v from (1.1), we can obtain the following dumped wave equation with nonlinear convection term:

$$u_{tt} - u_{xx} + u_t + f(u)_x = 0. \quad (1.2)$$

We consider the initial value problem for (1.2) with the initial conditions $u(0, x) = u_0(x)$, $u_t(0, x) = u_1(x)$. Our purpose is to show the global existence and asymptotic decay of solutions. Moreover, we show the asymptotic convergence of solutions toward nonlinear diffusion waves as $t \rightarrow \infty$. Orive-Zuazua [1] studied the case $f'(0) = 0$ and developed L^2 theory. We generalize their result and develop L^p theory in the general case where $|f'(0)| < 1$.

At first we state a result about the existence of global solutions in time.

Theorem 1.1 *Let $1 \leq p < \infty$ and assume that $u_0 \in L^1(\mathbb{R}) \cap W^{1,p}(\mathbb{R})$ and $u_1 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Put*

$$E_0 := \|u_0\|_{L^1} + \|u_0\|_{W^{1,p}} + \|u_1\|_{L^1} + \|u_1\|_{L^p}.$$

Then if E_0 is sufficiently small, then the initial problem for (1.2) has a unique global solution u with

$$u \in C([0, \infty); L^1(\mathbb{R}) \cap W^{1,p}(\mathbb{R})).$$

Moreover, the solution satisfies

$$\|u(t)\|_{L^q} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{q})} \quad (1 \leq \forall q \leq \infty), \quad (1.3)$$

$$\|u_x(t)\|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad (1.4)$$

where C is a constant.

Remark. When $p = \infty$, we have a similar global existence result in the space $L^\infty([0, \infty); L^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$.

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Nonlinear diffusion wave:

Next, we define a nonlinear diffusion wave which describes large-time behavior of solution constructed in Theorem 1.1. To this end, we apply the Chapman-Enskog expansion to the relaxation system (1.1). At the second order approximation of the expansion, we have the following viscous conservation law:

$$w_t + f(w)_x = (\mu(w)w_x)_x, \quad \mu(w) = 1 - (f'(w))^2. \quad (1.5)$$

It is known that the solution of such a viscous conservation law is asymptotically described by the nonlinear diffusion wave expressed in terms of the self-similar solution to the Burgers equation:

$$z_t + \left(\frac{z^2}{2}\right)_x = \mu z_{xx}, \quad \mu = \mu(0) = 1 - (f'(0))^2. \quad (1.6)$$

Therefore, it is expected that the solution of our equation (1.2) is also asymptotic to the same diffusion wave.

For the Burgers equation (1.6), the self-similar solution z is a solution which is invariant under the scale transformation $z^\lambda(t, x) := z(\lambda^2 t, \lambda x)$ ($\lambda > 0$), so that it takes the form $z = t^{-\frac{1}{2}} \phi(\frac{x}{\sqrt{t}})$. We denote by $z = Z(t, x; \mu, M)$ the self-similar solution for (1.6) which satisfies the integral condition $\int z dx = M$. This self-similar solution is given explicitly as

$$Z(t, x; \mu, M) = \sqrt{\frac{\mu}{t}} \frac{(e^{\frac{M}{2\mu}} - 1)e^{-y^2}}{\sqrt{\pi} + (e^{\frac{M}{2\mu}} - 1) \int_y^\infty e^{-\xi^2} d\xi}, \quad y = \frac{x}{\sqrt{4\mu t}}. \quad (1.7)$$

The nonlinear diffusion wave $w = W(t, x)$ for (1.2) is then defined by

$$W(t, x) = \beta^{-1} Z(t, x - \alpha t; \mu, \beta M), \quad M := \int (u_0 + u_1) dx, \quad (1.8)$$

where $\mu = 1 - (f'(0))^2$, $\alpha = f'(0)$, $\beta = f''(0)$. We note that this diffusion wave $w = W(t, x)$ has the conserved quantity $\int w dx = M = \int (u_0 + u_1) dx$ and satisfies

$$w_t + \left(\alpha w + \frac{\beta}{2} w^2\right)_x = \mu w_{xx}, \quad (1.9)$$

which is an approximation to (1.5).

We show that the global solution in Theorem 1.1 is asymptotic to the nonlinear diffusion wave defined above. We use the space L_1^1 which consists of functions f satisfying $(1 + |x|)f \in L^1$.

Theorem 1.2 *Let $1 \leq p < \infty$ and assume that $u_0 \in L_1^1(\mathbb{R}) \cap W^{1,p}(\mathbb{R})$ and $u_1 \in L_1^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Let $u(t, x)$ be the corresponding global solution of (1.2) constructed in Theorem 1.1, and let $W(t, x)$ be the nonlinear diffusion wave defined by (1.8). Put $w(t, x) = W(t + 1, x)$ and*

$$E_1 := \|u_0\|_{L_1^1} + \|u_0\|_{W^{1,p}} + \|u_1\|_{L_1^1} + \|u_1\|_{L^p}.$$

Then, for any $\epsilon > 0$, there is a positive constant δ_1 such that if $E_1 \leq \delta_1$, then we have the following asymptotic relations:

$$\|(u - w)(t)\|_{L^q} \leq C E_1 (1 + t)^{-\frac{1}{2}(1 - \frac{1}{q}) - \frac{1}{2} + \epsilon} \quad (1 \leq \forall q \leq \infty), \quad (1.10)$$

$$\|(u - w)_x(t)\|_{L^p} \leq C E_1 t^{-\frac{1}{2}} (1 + t)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2} + \epsilon}, \quad (1.11)$$

where C is a constant.

The key to the proof of these theorems is the derivation of detailed pointwise estimates of fundamental solutions to the linearized equation for (1.2). Pointwise estimates are derived by using the technique of Liu-Zeng [2], and the optimal decay estimates in L^p space are based on the pointwise estimates.

2 Fundamental solutions in Fourier space

We consider the linearized equation of (1.2):

$$u_{tt} - u_{xx} + u_t + \alpha u_x = 0, \quad (2.1)$$

with the initial conditions $u(0, x) = u_0(x)$, $u_t(0, x) = u_1(x)$. Taking the Fourier transform, we have

$$\hat{u}_{tt} + \hat{u}_t + (\xi^2 + \alpha i \xi) \hat{u} = 0, \quad (2.2)$$

with the corresponding initial conditions $\hat{u}(0, \xi) = \hat{u}_0(\xi)$, $\hat{u}_t(0, \xi) = \hat{u}_1(\xi)$. The characteristic equation of (2.2) becomes $\lambda^2 + \lambda + (\xi^2 + \alpha i \xi) = 0$, and the eigenvalues are given by $\lambda_1(\xi) = \frac{1}{2}(-1 + \sqrt{1 - 4(\xi^2 + \alpha i \xi)})$, $\lambda_2(\xi) = \frac{1}{2}(-1 - \sqrt{1 - 4(\xi^2 + \alpha i \xi)})$. Then the solution of (2.2) can be expressed as

$$\hat{u}(t, \xi) = \hat{G}(t, \xi)(\hat{u}_0(\xi) + \hat{u}_1(\xi)) + \hat{H}(t, \xi)\hat{u}_0(\xi), \quad (2.3)$$

where

$$\hat{G}(t, \xi) = \frac{1}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t}), \quad \hat{H}(t, \xi) = \frac{1}{\lambda_1 - \lambda_2}((1 + \lambda_1)e^{\lambda_2 t} - (1 + \lambda_2)e^{\lambda_1 t}).$$

The functions $G(t, x)$ and $H(t, x)$ are called fundamental solutions of (2.1).

We give asymptotic expressions of $\hat{G}(t, \xi)$ for $|\xi| \rightarrow 0$ and $|\xi| \rightarrow \infty$. Similar expressions for $\hat{H}(t, \xi)$ are omitted here.

Case 1. In the low frequency region where $\xi \in \mathbb{C}$ and $|\xi| \rightarrow 0$, we have an expression $\hat{G} = \hat{G}_0 + \hat{R}_0$ with

$$\hat{G}_0 = e^{-(\alpha i \xi + \mu \xi^2)t}, \quad \hat{R}_0 = e^{-(\alpha i \xi + \mu \xi^2)t} \hat{R}_{0,1} + e^{-t} \hat{R}_{0,2},$$

where

$$|\hat{R}_{0,1}| \leq C|\xi|(|\xi|^2 t + 1)e^{C|\xi|^3 t}, \quad |\hat{R}_{0,2}| \leq C e^{C|\xi|t}.$$

Also, we have another expression of the form

$$\hat{G} = e^{-(\alpha i \xi + \mu \xi^2)t} \hat{G}_1 + e^{-t} \hat{G}_2, \quad |\hat{G}_1| \leq C e^{C|\xi|^3 t}, \quad |\hat{G}_2| \leq C e^{C|\xi|t}.$$

Case 2. In the high frequency region where $\xi \in \mathbb{C}$ and $|\xi| \rightarrow \infty$, we have an expression $\hat{G} = \hat{G}_\infty^{(l)} + \hat{R}_\infty^{(l)}$ with $\hat{G}_\infty^{(l)} = 0$ ($l = 0$) and

$$\begin{aligned} \hat{G}_\infty^{(l)} &= \sum_{k=0}^{l-1} \{e^{-(\kappa - i\xi)t} P_k(t) + e^{-(\nu + i\xi)t} Q_k(t)\} (i\xi)^{-k-1} \quad (l \geq 1), \\ \hat{R}_\infty^{(l)} &= \{e^{-(\kappa - i\xi)t} P_l(t) + e^{-(\nu + i\xi)t} Q_l(t)\} (i\xi)^{-l-1} + e^{-(\kappa - i\xi)t} \hat{R}_{\infty,1}^{(l)} + e^{-(\nu + i\xi)t} \hat{R}_{\infty,2}^{(l)} \quad (l \geq 0), \end{aligned}$$

where $P_k(t)$, $Q_k(t)$ are polynomials of t with degree k and

$$|\hat{R}_{\infty,1}^{(l)}| + |\hat{R}_{\infty,2}^{(l)}| \leq C|\xi|^{-l-2}(1+t)^{l+1} e^{C|\xi|^{-1}t}.$$

3 Fundamental solutions

We give expressions of the fundamental solution $G(t, x)$ with detailed pointwise estimates.

Proposition 3.1 G can be expressed as $G = G_0 + G_\infty^{(l)} + R^{(l)} = G_\infty^{(l)} + R_\infty^{(l)}$. Here

$$G_0 = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{(x-\alpha t)^2}{4\mu t}}, \quad G_\infty^{(l)} = 0 \quad (l = 0),$$

$$\partial_x^l G_\infty^{(l)} = \sum_{k=0}^{l-1} \{e^{-\kappa t} P_k(t) \partial_x^{l-k-1} \delta(x+t) + e^{-\nu t} Q_k(t) \partial_x^{l-k-1} \delta(x-t)\} \quad (l \geq 1),$$

where $P_k(t)$, $Q_k(t)$ are the polynomials of t appeared in the previous section and δ is the Dirac delta function. The remainder terms satisfy the following pointwise estimates:

$$|\partial_x^l R^{(l)}| \leq C(1+t)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} e^{-c\frac{(x-\alpha t)^2}{t}} + C e^{-c(t+|x|)},$$

$$|\partial_x^l R_\infty^{(l)}| \leq C(1+t)^{-\frac{l+1}{2}} e^{-c\frac{(x-\alpha t)^2}{t}} + C e^{-c(t+|x|)}.$$

This result implies that the fundamental solution G can be well approximated by the heat kernel G_0 as $t \rightarrow \infty$. We have a similar expression also for H .

Proposition 3.2 We can express H as $H = H_\infty^{(l)} + S_\infty^{(l)}$. Here

$$\partial_x^l H_\infty^{(l)} = \sum_{k=0}^l \{e^{-\kappa t} \tilde{P}_k(t) \partial_x^{l-k} \delta(x+t) + e^{-\nu t} \tilde{Q}_k(t) \partial_x^{l-k} \delta(x-t)\},$$

where $\tilde{P}_k(t)$, $\tilde{Q}_k(t)$ are some polynomials of t with degree k and δ denotes the Dirac delta function. The remainder term satisfies the following pointwise estimate:

$$|\partial_x^l S_\infty^{(l)}| \leq C(1+t)^{-\frac{l+2}{2}} e^{-c\frac{(x-\alpha t)^2}{t}} + C e^{-c(t+|x|)}.$$

As a corollary of the above pointwise estimates of the fundamental solutions, we have the following $L^p - L^q$ estimates for solutions to the linearized equation (2.1).

Proposition 3.3 Let $1 \leq q \leq p \leq \infty$. Then we have the following $L^p - L^q$ estimates:

$$\|\partial_x^l G(t) * f\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l}{2}} \|f\|_{L^q} + C e^{-ct} \|f\|_{W^{l-1,p}} \quad (l \geq 1),$$

$$\|\partial_x^l G(t) * f\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q} \quad (l = 0),$$

$$\|\partial_x^l H(t) * f\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l+1}{2}} \|f\|_{L^q} + C e^{-ct} \|f\|_{W^{l,p}} \quad (l \geq 0).$$

Moreover, G is approximated by G_0 in the following sense:

$$\|\partial_x^l (G - G_0)(t) * f\|_{L^p} \leq C t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l}{2}} (1+t)^{-\frac{l}{2}} \|f\|_{L^q} + C e^{-ct} \|f\|_{W^{l-1,p}} \quad (l \geq 1),$$

$$\|\partial_x^l (G - G_0)(t) * f\|_{L^p} \leq C t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} (1+t)^{-\frac{l}{2}} \|f\|_{L^q} \quad (l = 0).$$

4 Outline of proof of Theorem 1.1

By the Duhamel principle, we see that the solution to the nonlinear equation (1.2) satisfies

$$u(t) = G(t) * (u_0 + u_1) + H(t) * u_0 - \int_0^t G(t-s) * g(u)_x(s) ds, \quad (4.1)$$

where $g(u) := f(u) - f(0) - f'(0)u$. We introduce a mapping Φ by

$$\Phi[u](t) := G(t) * (u_0 + u_1) + H(t) * u_0 - \int_0^t G(t-s) * g(u)_x(s) ds, \quad (4.2)$$

and solve the integral equation (4.1) by applying the contraction mapping principle. To this end, we consider the Banach space $X := C([0, \infty); L^1 \cap W^{1,p})$ ($1 \leq p < \infty$) with the norm

$$\|u\|_X := \sup_{0 \leq s \leq t} \left\{ \|u\|_{L^1} + (1+s)^{\frac{1}{2}} \|u\|_{L^\infty} + (1+s)^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}} \|\partial_x u\|_{L^p} \right\},$$

and also a closed convex subset $S := \{u \in X; \|u\|_X \leq 2C_0 E_0\}$. Then, applying the $L^p - L^q$ estimates in Proposition 3.3 to (4.2), we have:

Claim: There are positive constants δ_0 and C_0 such that if $E_0 \leq \delta_0$, then Φ is a contraction mapping of S into S .

Consequently, we find a unique fixed point $u \in S$ of Φ , and this fixed point is the desired global solution to (1.2).

5 Outline of proof of Theorem 1.2

The diffusion wave w solves (1.9) and hence the integral equation

$$w(t) = G_0(t) * w_0 - \frac{\beta}{2} \int_0^t G_0(t-s) * (w^2)_x(s) ds, \quad (5.1)$$

where $w_0(x) = W(1, x)$. Subtract (5.1) from (4.1), we have

$$\begin{aligned} u(t) - w(t) &= \{G(t) - G_0(t)\} * (u_0 + u_1) + G_0(t) * (u_0 + u_1 - w_0) \\ &\quad + H(t) * u_0 - \frac{\beta}{2} \int_0^t \{G(t-s) - G_0(t-s)\} * (u^2)_x(s) ds \\ &\quad - \frac{\beta}{2} \int_0^t G_0(t-s) * (u^2 - w^2)_x(s) ds - \int_0^t G(t-s) * \left\{g(u) - \frac{\beta}{2} u^2\right\}_x(s) ds. \end{aligned} \quad (5.2)$$

We can show the desired inequalities (1.10) and (1.11) by estimating (5.2) in the same way as in the proof of Theorem 1.1.

References

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