

# AN INTRODUCTION OF GENERALIZED HEEGAARD SPLITTINGS

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## 1. INTRODUCTION

This article is based on a lecture series given by Scharlemann and Schultens at RIMS in the summer of 2001. The purpose of the lecture series lay in familiarizing the audience with the basics of three-dimensional manifold theory and introducing some topics of current research.

A way to study (connected compact orientable) three-dimensional manifolds, 3-manifolds briefly, is to decompose a 3-manifold into two ‘elementary’ manifolds, so-called compression bodies. Such a decomposition is called a Heegaard splitting. It is known that there are many correlations between Heegaard splittings and topology of 3-manifolds.

Roughly, a generalized Heegaard splitting is a decomposition of a 3-manifold into more than two compression bodies.

If these topics should happen to be of interest, please see our lecture notes [5].

## 2. PRELIMINARIES

The notation  $\eta(B; A)$  denotes a regular neighborhood of  $B$  in  $A$ . By the term *surface*, we will mean a connected compact 2-manifold. Let  $F$  be a surface. A loop  $\alpha$  in  $F$  is said to be *inessential* in  $F$  if  $\alpha$  bounds a disk in  $F$ , otherwise  $\alpha$  is said to be *essential* in  $F$ .

We always let  $M$  be a connected compact orientable 3-manifold. A surface  $F \subset M$  is said to be *properly* embedded in  $M$  if  $F$  is embedded in  $M$  such that  $F \cap \partial M = \partial F$ . A disk  $D$  properly embedded in  $M$  is said to be *inessential* in  $M$  if  $D$  cuts off a 3-ball from  $M$ , otherwise  $D$  is said to be *essential* in  $M$ . A 2-sphere  $P$  properly embedded in  $M$  is said to be *inessential* in  $M$  if  $P$  bounds a 3-ball in  $M$  or cuts off  $P \times [0, 1]$  from  $M$ , otherwise  $P$  is said to be *essential* in  $M$ . Let  $F$  be a surface properly embedded in  $M$ . We say that  $F$  is *compressible* in  $M$  if there is a disk  $D \subset M$  such that  $D \cap F = \partial D$  and  $\partial D$  is an essential loop in  $F$ . We say that  $F$  is *incompressible* in  $M$  if  $F$  is not compressible in  $M$ . Suppose that  $F$  is homeomorphic neither to a disk nor to a 2-sphere. The surface  $F$  is said to be *essential* in  $M$  if  $F$  is incompressible in  $M$  and does not cut off  $F \times [0, 1]$  from  $M$ .

**Definition 2.1.** Let  $M$  be a connected compact orientable 3-manifold.

- (1)  $M$  is said to be *reducible* if  $M$  contains an essential 2-sphere.  $M$  is said to be *irreducible* if  $M$  is not reducible.
- (2)  $M$  is said to be *toroidal* if  $M$  contains an essential torus.

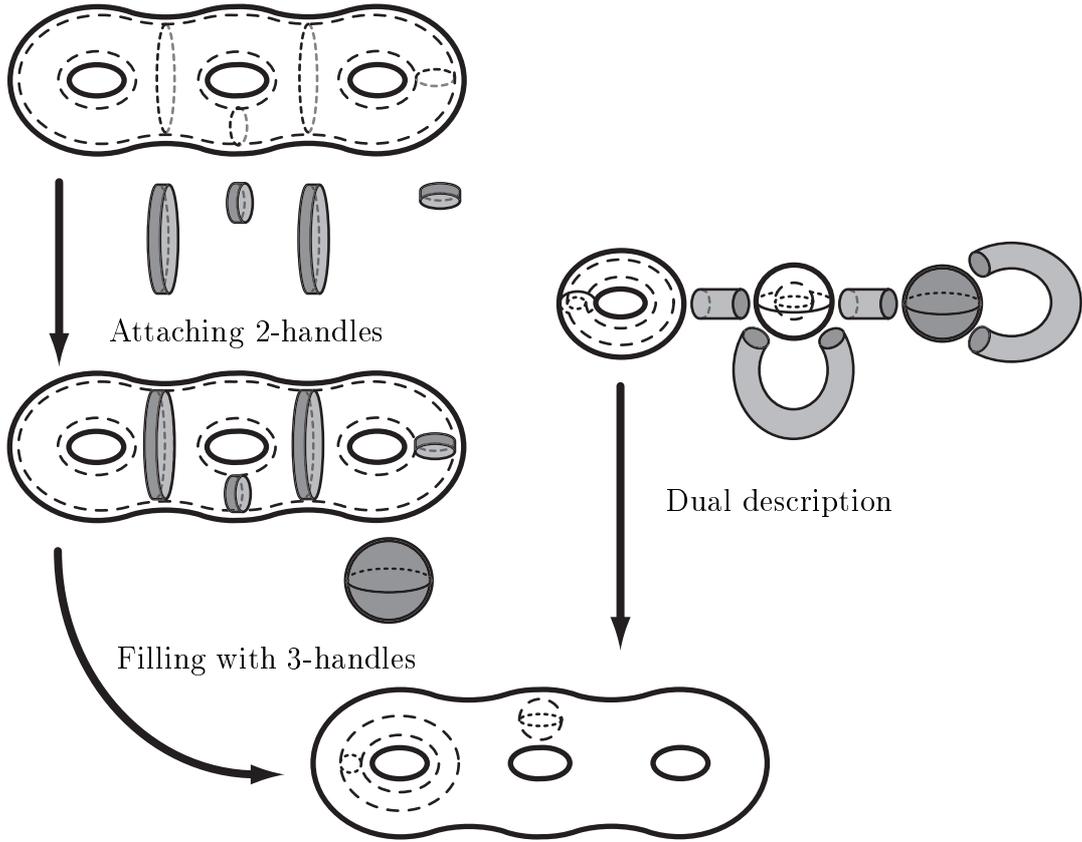


FIGURE 1. Construction of a compression body

### 3. HEEGAARD SPLITTINGS

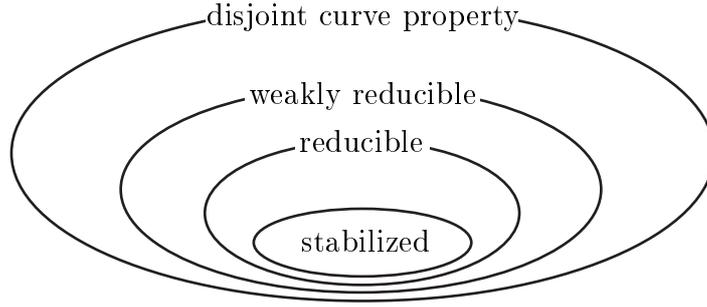
**Definition 3.1.** A 3-manifold  $C$  is called a *compression body* if there exists a closed surface  $F$  such that  $C$  is obtained from  $F \times [0, 1]$  by attaching 2-handles along mutually disjoint loops in  $F \times \{1\}$  and filling in some resulting 2-sphere boundary components with 3-handles. We denote  $F \times \{0\}$  by  $\partial_+ C$  and  $\partial C \setminus \partial_+ C$  by  $\partial_- C$ . A compression body  $C$  is called a *handlebody* if  $\partial_- C = \emptyset$ .

A compression body  $C$  is also obtained from  $\partial_- C \times [0, 1]$  and some (possibly empty) 3-balls by attaching some 1-handles to  $\partial_- C \times \{1\}$  and the boundary of the 3-balls. This is called a *dual description* of compression bodies (cf. Figure 1).

**Definition 3.2.** Let  $(\partial_1 M, \partial_2 M)$  be a partition of  $\partial$ -components of  $M$ . A triplet  $(C_1, C_2; S)$  is called a *Heegaard splitting* of  $(M; \partial_1 M, \partial_2 M)$  if  $C_1$  and  $C_2$  are compression bodies with  $C_1 \cup C_2 = M$ ,  $\partial_- C_1 = \partial_1 M$ ,  $\partial_- C_2 = \partial_2 M$  and  $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = S$ . The surface  $S$  is called a *Heegaard surface* and the *genus* of a Heegaard splitting is defined by the genus of the Heegaard surface.

**Theorem 3.3** (cf. Moise [4]). *For any partition  $(\partial_1 M, \partial_2 M)$  of the boundary components of  $M$ , there is a Heegaard splitting of  $(M; \partial_1 M, \partial_2 M)$ .*

Let  $(C_1, C_2; S)$  be a Heegaard splitting of  $(M; \partial_1 M, \partial_2 M)$ . By a dual description of  $C_1$ , we see that  $C_1$  is obtained from  $\partial_1 M \times [0, 1]$  and 0-handles  $\mathcal{H}^0$  by attaching 1-handles  $\mathcal{H}^1$ . By Definition 3.1,  $C_2$  is obtained from  $S \times [0, 1]$  by attaching 2-handles

FIGURE 2. Inclusion relation in case of  $\text{genus}(S) \geq 2$ .

$\mathcal{H}^2$  and filling some 2-sphere boundary components with 3-handles  $\mathcal{H}^3$ . Hence we obtain the following decomposition of  $M$ :

$$M = \partial_1 M \times [0, 1] \cup \mathcal{H}^0 \cup \mathcal{H}^1 \cup S \times [0, 1] \cup \mathcal{H}^2 \cup \mathcal{H}^3.$$

By collapsing  $S \times [0, 1]$  to  $S$ , we have:

$$M = \partial_1 M \times [0, 1] \cup \mathcal{H}^0 \cup \mathcal{H}^1 \cup_S \mathcal{H}^2 \cup \mathcal{H}^3.$$

This is called a *handle decomposition of  $M$  induced from  $(C_1, C_2; S)$* .

**Definition 3.4.** Let  $(C_1, C_2; S)$  be a Heegaard splitting of  $(M; \partial_1 M, \partial_2 M)$ .

- (1) The splitting  $(C_1, C_2; S)$  is said to be *stabilized* if there are essential disks  $D_i$  ( $i = 1, 2$ ) of  $C_i$  such that  $\partial D_1$  and  $\partial D_2$  intersect transversely in a single point.
- (2) The splitting  $(C_1, C_2; S)$  is said to be *reducible* if there are essential disks  $D_i$  ( $i = 1, 2$ ) of  $C_i$  with  $\partial D_1 = \partial D_2$ . The splitting  $(C_1, C_2; S)$  is said to be *irreducible* if  $(C_1, C_2; S)$  is not reducible.
- (3) The splitting  $(C_1, C_2; S)$  is said to be *weakly reducible* if there are essential disks  $D_i$  ( $i = 1, 2$ ) of  $C_i$  with  $\partial D_1 \cap \partial D_2 = \emptyset$ . The splitting  $(C_1, C_2; S)$  is said to be *strongly irreducible* if  $(C_1, C_2; S)$  is not *weakly reducible*.
- (4) The splitting  $(C_1, C_2; S)$  admits *disjoint curve property* if there are essential disks  $D_i$  ( $i = 1, 2$ ) of  $C_i$  and an essential loop  $z$  with  $(\partial D_1 \cup \partial D_2) \cap z = \emptyset$ .

Suppose that  $(C_1, C_2; S)$  is stabilized, and let  $D_i$  ( $i = 1, 2$ ) be disks as in (1) of Definition 3.4. Note that since  $\partial D_1$  intersects  $\partial D_2$  transversely in a single point, we see that each of  $\partial D_i$  ( $i = 1, 2$ ) is non-separating in  $S$  and hence each of  $D_i$  ( $i = 1, 2$ ) is non-separating in  $C_i$ . Set  $C'_1 = \text{cl}(C_1 \setminus \eta(D_1; C_1))$  and  $C'_2 = C_2 \cup \eta(D_1; C_1)$ . Then each of  $C'_i$  ( $i = 1, 2$ ) is a compression body with  $\partial_+ C'_1 = \partial_+ C'_2$ . Set  $S' = \partial_+ C'_1 (= \partial_+ C'_2)$ . Then we obtain the Heegaard splitting  $(C'_1, C'_2; S')$  of  $M$  with  $\text{genus}(S') = \text{genus}(S) - 1$ . Conversely,  $(C_1, C_2; S)$  is obtained from  $(C'_1, C'_2; S')$  by adding a trivial handle. We say that  $(C_1, C_2; S)$  is obtained from  $(C'_1, C'_2; S')$  by *stabilization*. Figure 2 describes inclusion relation among the properties in Definition 3.4.

The following is part of relation between Heegaard splittings and topology of 3-manifolds. For simplicity, we here assume  $\partial M = \emptyset$ .

**Theorem 3.5.** *Suppose that  $\partial M = \emptyset$ .*

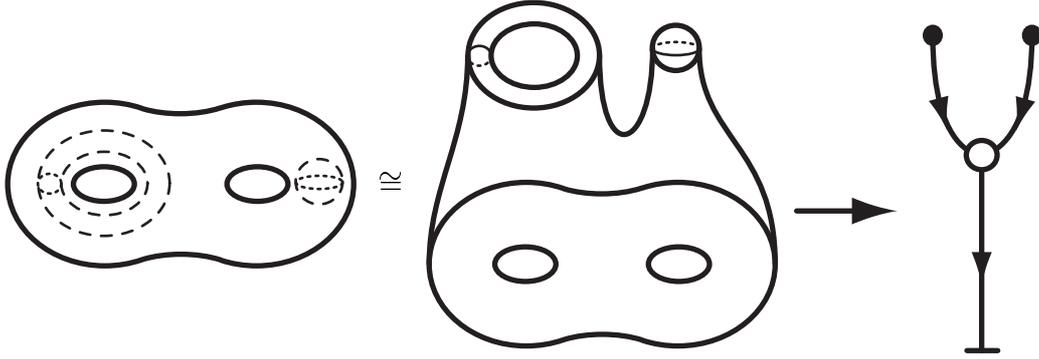


FIGURE 3. The middle is a schematic of a compression body on the left side.

- (1) (Haken [2]) *Let  $(C_1, C_2; S)$  be a Heegaard splitting of  $(M; \partial_1 M, \partial_2 M)$ . If  $M$  is reducible, then  $(C_1, C_2; S)$  is reducible.*
- (2) (Casson-Gordon [1]) *Let  $(C_1, C_2; S)$  be a Heegaard splitting of  $(M; \partial_1 M, \partial_2 M)$ . If  $(C_1, C_2; S)$  is weakly reducible and is not reducible, then  $M$  contains a closed incompressible surface.*
- (3) (Hempel [3]) *Let  $(C_1, C_2; S)$  be a Heegaard splitting of  $(M; \partial_1 M, \partial_2 M)$  with  $\text{genus}(S) \geq 2$ . If  $M$  is toroidal or is a Seifert fibered manifold, then  $(C_1, C_2; S)$  admits disjoint curve property.*

#### 4. GENERALIZED HEEGAARD SPLITTINGS

Collapsing a compression body  $C$ , we can say that it looks like a *fork*. The grip of a fork corresponds to  $\partial_+ C$ , and each tine just corresponds to a component of  $\partial_- C$ . This idea is so convenient for defining generalized Heegaard splittings.

**Definition 4.1.** A *0-fork* is a connected 1-complex obtained by joining a point  $p$  to a point  $g$  whose 1-simplexes are oriented toward  $g$  and away from  $p$ . For  $n \geq 1$ , an *n-fork* is a connected 1-complex obtained by joining a point  $p$  to each of distinct  $n$  points  $t_i$  ( $i = 1, \dots, n$ ) and to a point  $g$  whose 1-simplexes are oriented toward  $g$  and away from  $t_i$ . We call  $p$  a *root*,  $t_i$  a *tine* and  $g$  a *grip*.

As mentioned previously, an  $n$ -fork corresponds to a compression body  $C$  such that each of  $t_i$  ( $i = 1, 2, \dots, n$ ) corresponds to a component of  $\partial_- C$  and  $g$  corresponds to  $\partial_+ C$  (cf. Figure 3).

**Definition 4.2.** Let  $\mathcal{A}$  ( $\mathcal{B}$  resp.) be a collection of finite forks,  $T_{\mathcal{A}}$  ( $T_{\mathcal{B}}$  resp.) a collection of tines of  $\mathcal{A}$  ( $\mathcal{B}$  resp.) and  $G_{\mathcal{A}}$  ( $G_{\mathcal{B}}$  resp.) a collection of grips of  $\mathcal{A}$  ( $\mathcal{B}$  resp.). We suppose that there are bijections  $\mathcal{T} : T_{\mathcal{A}} \rightarrow T_{\mathcal{B}}$  and  $\mathcal{G} : G_{\mathcal{A}} \rightarrow G_{\mathcal{B}}$ . A *fork complex*  $\mathcal{F}$  is an oriented connected 1-complex  $\mathcal{A} \cup (-\mathcal{B}) / \{\mathcal{T}, \mathcal{G}\}$ , where  $-\mathcal{B}$  denotes the 1-complex obtained by taking the opposite orientation of each 1-simplex and the equivalence relation  $/\{\mathcal{T}, \mathcal{G}\}$  is given by  $t \sim \mathcal{T}(t)$  for any  $t \in T_{\mathcal{A}}$  and  $g \sim \mathcal{G}(g)$  for any  $g \in G_{\mathcal{A}}$ . We define:

$$\begin{aligned} \partial_1 \mathcal{F} &= \{(\text{the tines of } \mathcal{A}) \setminus T_{\mathcal{A}}\} \cup \{(\text{the grips of } \mathcal{B}) \setminus G_{\mathcal{B}}\} \text{ and} \\ \partial_2 \mathcal{F} &= \{(\text{the tines of } \mathcal{B}) \setminus T_{\mathcal{B}}\} \cup \{(\text{the grips of } \mathcal{A}) \setminus G_{\mathcal{A}}\}. \end{aligned}$$

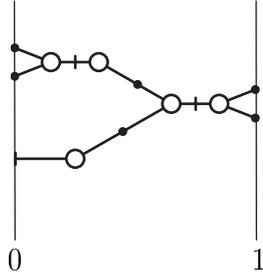


FIGURE 4. An example of an exact fork complex.

**Definition 4.3.** A fork complex  $\mathcal{F}$  is *exact* if we can put  $\mathcal{F}$  in  $\mathbb{R}^3$  so that

- (1)  $\partial_1\mathcal{F}$  lies in the plane of height 0,
- (2) for any simple path  $\alpha$  in  $\mathcal{F}$  from a point in  $\partial_1\mathcal{F}$  to a point in  $\partial_2\mathcal{F}$ ,  $h|_\alpha$  is monotonically increasing, where  $h$  is the height function of  $\mathbb{R}^3$  and
- (3)  $\partial_2\mathcal{F}$  lies in the plane of height 1 (cf. Figure 4).

In the following, we regard fork complexes as geometric objects, i.e., 1-dimensional polyhedra.

**Definition 4.4.** A *fork* of  $\mathcal{F}$  is the image of a fork in  $\mathcal{A} \cup \mathcal{B}$  in  $\mathcal{F}$ . A *grip* (root and *tine* resp.) of  $\mathcal{F}$  is the image of a grip (root and *tine* resp.) in  $\mathcal{A} \cup \mathcal{B}$  in  $\mathcal{F}$ .

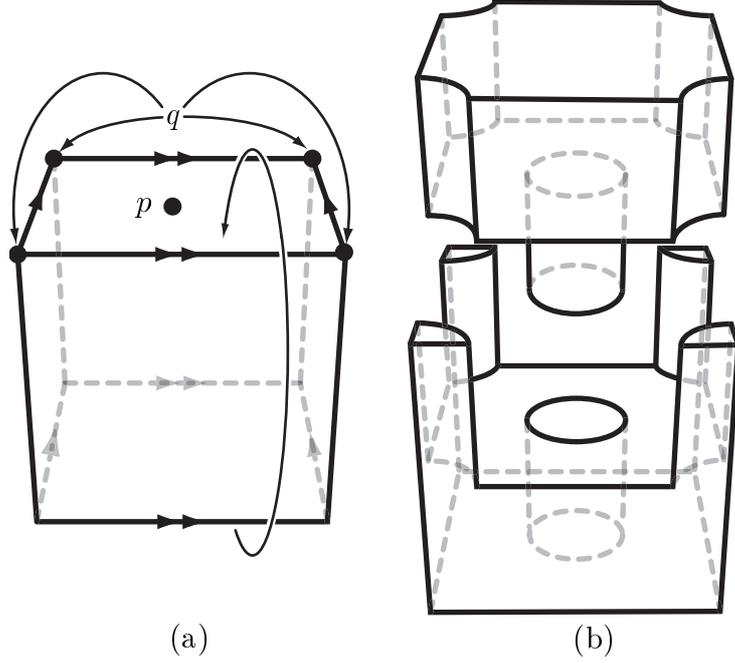
**Definition 4.5.** Let  $M$  be a compact orientable 3-manifold, and let  $(\partial_1M, \partial_2M)$  be a partition of boundary components of  $M$ . A *generalized Heegaard splitting* of  $(M; \partial_1M, \partial_2M)$  is a pair of an exact fork complex  $\mathcal{F}$  and a proper map  $\rho : (M; \partial_1M, \partial_2M) \rightarrow (\mathcal{F}; \partial_1\mathcal{F}, \partial_2\mathcal{F})$  which satisfies the following.

- (1) The map  $\rho$  is transverse to  $\mathcal{F} - \{\text{the roots of } \mathcal{F}\}$ .
- (2) For each fork  $\mathcal{F} \subset \mathcal{F}$ , we have the following.
  - (a) If  $\mathcal{F}$  is a 0-fork, then  $\rho^{-1}(\mathcal{F})$  is a handlebody  $V_{\mathcal{F}}$  such that (1)  $\rho^{-1}(g) = \partial V_{\mathcal{F}}$  and (2)  $\rho^{-1}(p)$  is a 1-complex which is a spine of  $V_{\mathcal{F}}$ , where  $g$  is the grip of  $\mathcal{F}$ .
  - (b) If  $\mathcal{F}$  is an  $n$ -fork with  $n \geq 1$ , then  $\rho^{-1}(\mathcal{F})$  is a connected compression body  $V_{\mathcal{F}}$  such that (1)  $\rho^{-1}(g) = \partial_+ V_{\mathcal{F}}$ , (2) for each *tine*  $t_i$ ,  $\rho^{-1}(t_i)$  is a connected component of  $\partial_- V_{\mathcal{F}}$  and  $\rho^{-1}(t_i) \neq \rho^{-1}(t_j)$  for  $i \neq j$  and (3)  $\rho^{-1}(p)$  is a 2-complex which is a deformation retract of  $V_{\mathcal{F}}$ , where  $g$  is the grip of  $\mathcal{F}$ ,  $p$  is the root of  $\mathcal{F}$  and  $\{t_i\}_{1 \leq i \leq n}$  is the set of the *tines* of  $\mathcal{F}$ .

Let  $g$  be a grip of  $\mathcal{F}$  which is contained in the interior of  $\mathcal{F}$ . Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be the forks of  $\mathcal{F}$  which are adjacent to  $g$ . Then  $(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$  is a Heegaard splitting of  $\rho^{-1}(\mathcal{F}_1 \cup \mathcal{F}_2)$ . Moreover, if it is weakly reducible, then we can obtain another generalized Heegaard splitting via *weak reduction*. In the next section, we demonstrate a concrete example of the way weak reduction is used.

## 5. EXAMPLES OF GENERALIZED HEEGAARD SPLITTINGS

In this section, we obtain a generalized Heegaard splitting and associated fork complex of the 3-torus  $T^3 = S^1 \times S^1 \times S^1$ . It is known that  $T^3$  is obtained from a

FIGURE 5. A Heegaard splitting of  $T^3$ .

cube  $[0, 1] \times [0, 1] \times [0, 1]$  by attaching corresponding edges and faces as in Figure 5 (a). Set

$$A = \text{cl}((T^2 \times [0, 1/2]) \setminus \eta(p \times [0, 1/2]; T^2 \times [0, 1/2])) \cup \eta(q \times [1/2, 1]; T^2 \times [1/2, 1])$$

and

$$\begin{aligned} B &= \text{cl}(T^3 \setminus A) \\ &= \text{cl}((T^2 \times [1/2, 1]) \setminus \eta(q \times [1/2, 1]; T^2 \times [1/2, 1])) \cup \eta(p \times [0, 1/2]; T^2 \times [0, 1/2]). \end{aligned}$$

Then we see that  $A$  and  $B$  are genus two handlebodies and hence  $(A, B; S)$  is a Heegaard splitting of  $T^3$ , where  $S = \partial A = \partial B$  (cf. Figure 5 (b)). Set  $h^1 = \eta(q \times [1/2, 1]; T^2 \times [1/2, 1])$  and  $h^2 = \eta(p \times [0, 1/2]; T^2 \times [0, 1/2])$ . Note that  $h^1$  ( $h^2$  resp.) can be regarded as a 1-handle (2-handle resp.) in a handle decomposition of  $T^3$  obtained from the Heegaard splitting  $(A, B; S)$ . Since  $h^1 \cap h^2 = \emptyset$ , we can perform a *weak reduction* to obtain a generalized Heegaard splitting. We give a concrete description of the generalized Heegaard splitting in the following. First, set  $A_1 = \text{cl}(T^3 \times [0, 1/2] \setminus h^1)$  and  $B_2 = \text{cl}(T^3 \times [1/2, 1] \setminus h^2)$ . That is,  $A_1$  is obtained from  $A$  by removing the 1-handle  $h^1$  and  $B_2$  is obtained from  $B$  by removing the 2-handle  $h^2$ . Then we have:

$$\begin{aligned} T^3 &= A \cup B \\ &= A_1 \cup h^1 \cup h^2 \cup B_2 \\ &\cong A_1 \cup (\partial A_1 \times [0, 1]) \cup h^1 \cup h^2 \cup B_2 \\ &= A_1 \cup ((\partial A_1 \times [0, 1]) \cup h^2) \cup h^1 \cup B_2. \end{aligned}$$

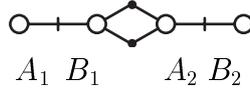


FIGURE 6. The associated fork complex.

Set  $B_1 = (\partial A_1 \times [0, 1]) \cup h^2$ . Then  $B_1$  is a compression body such that  $\partial_+ B_1 = \partial A_1$  and  $\partial_- B_1$  consists of two tori. Hence we have:

$$\begin{aligned} T^3 &\cong A_1 \cup B_1 \cup h^1 \cup B_2 \\ &\cong A_1 \cup B_1 \cup ((\partial_- B_1 \times [0, 1]) \cup h^1) \cup B_2. \end{aligned}$$

Set  $A_2 = (\partial_- B_1 \times [0, 1]) \cup h^1$ . Then  $A_2$  is a compression body such that  $\partial_+ A_2 = \partial B_1$  and  $\partial_- A_2 = \partial_- B_1$ . Hence we have:

$$T^3 = (A_1 \cup B_1) \cup (A_2 \cup B_2).$$

This together with the fork complex as in Figure 6 gives a generalized Heegaard splitting.

## REFERENCES

- [1] A. Casson and C. McA. Gordon, *Reducing Heegaard splittings*, Topology Appl. **27** (1987), 275-283.
- [2] W. Haken, *Some results on surfaces in 3-manifolds*, Studies in Modern Topology, Math. Assoc. Amer. 1968, 39-98.
- [3] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology, **40** (2001), 631-657.
- [4] E. Moise, *Affine structures in 3-manifolds whose homeomorphism classes of Heegaard splittings have polynomial growth*, Ann. of Math. **55** (1952), 96-114.
- [5] T. Saito, M. Scharlemann and J. Schultens, *Lecture notes on generalized Heegaard splittings*, preprint, ArXiv: math.GT/0504167.
- [6] J. Schultens, *The classification of Heegaard splittings for (compact orientable surfaces)  $\times S^1$* , Proc. London Math. Soc. **67** (1993), 425-448.

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