The stability of Coxeter type arrangements

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0 Introduction

A *hyperplane arrangement* (or simply an *arrangement*) is a finite collection of affine hyperplanes in a fixed vector space V over a field $\mathbb{K}$. This is a very simple geometric object, but there are a lot of interesting problems on arrangements. Especially, one of the most interesting problems is that called Terao conjecture, which asserts some algebraic structure associated to an arrangement depends only on the combinatorial characterization of the arrangement. In this abstract, we introduce some basic definitions, results and concepts of the hyperplane arrangement theory to explain what Terao conjecture is. Moreover, we show the results on the stability of some arrangements which give a new way to consider Terao conjecture.

1 Hyperplane arrangements

In this section we introduce some elementary definitions and results on hyperplane arrangements, for which we refer the reader to [OT].

1.1 General Definition

Let $\mathbb{K}$ be a field of any characteristic and V be an l-dimensional vector space over $\mathbb{K}$. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in V. We say an arrangement $\mathcal{A}$ is *central* if each hyperplane $H \in \mathcal{A}$ is a vector subspace of V, and *essential* if $\bigcap_{H\in\mathcal{A}} H$ is the origin. An arrangement in an l-dimensional vector space is called an *l-arrangement*.

1.2 Combinatorics of arrangements

We define two polynomials associated to an arrangement, which contain a lot of information of arrangements and its complements. To define them, we introduce the *intersection lattice* of an arrangement.

Definition 1.1
Let $\mathcal{A}$ be an arrangement in a fixed vector space V of dimension l. The intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$L(\mathcal{A}) := \{\bigcap_{H\in\mathcal{B}} H \neq \emptyset \ |\mathcal{B} \subset \mathcal{A}\},$$

where we agree $\bigcap_{H\in\mathcal{B}} H = V$ when $\mathcal{B} = \emptyset$.

The intersection lattice contains all the combinatorial information of an arrangement. We introduce the partial order in $L(\mathcal{A})$ by the following manner:

$$\text{For } X,\ Y \in L(\mathcal{A}),\ X \geq Y \iff X \subseteq Y.$$

Next we introduce the Möbius function on $L(\mathcal{A})$.

Definition 1.2
For an arrangement $\mathcal{A}$ and its intersection lattice $L(\mathcal{A})$, the Möbius function μ from $L(\mathcal{A})$ to $\mathbb{Z}$ is defined as follows:

$$\begin{aligned} \mu(V) &= 1, \\ \sum_{V\leq Y\leq X} \mu(Y) &= 0 \text{ for } V \neq X \in L(\mathcal{A}). \end{aligned}$$

By using these concepts, we can define two important polynomials.

Definition 1.3
The characteristic polynomial of an l-arrangement $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) := \sum_{X\in L(\mathcal{A})} \mu(X) t^{\dim X},$$

and the Poincaré polynomial $\pi(\mathcal{A}, t)$ by

$$\pi(\mathcal{A}, t) := \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{codim} X}.$$

It is obvious that

$$\chi(\mathcal{A},\, t) = t^l \pi(\mathcal{A},\, -1/t).$$

There are a lot of interesting relations between these polynomials and algebra, topology of arrangements. For example, we can see the number of chambers as follows:

Theorem 1.1 (Zaslavsky)
Let $\mathbb{K}$ be a real number field, $\mathcal{A}$ be an arrangement and $\mathcal{C}(\mathcal{A})$ be the set of chambers in $V \setminus \bigcup_{H \in \mathcal{A}} H$. Then

$$\sharp\mathcal{C}(\mathcal{A}) = \pi(\mathcal{A}, 1).$$

1.3 Logarithmic vector fields

Now we define the module of logarithmic vector fields which we are the most interested in. Let $\{X_1, \ldots, X_l\}$ be a basis for the dual vector space V^*, S be a symmetric algebra of V^* (hence $S \simeq \mathbb{K}[X_1, \ldots, X_l]$) and $\mathrm{Der}_{\mathbb{K}}(S)$ be the set of $\mathbb{K}$-linear derivations of S. For an arrangement $\mathcal{A}$ and each hyperplane $H \in \mathcal{A}$, fix an element $\alpha_H \in S$ such that $\ker(\alpha_H) = H$.

Definition 1.4
For an arrangement $\mathcal{A}$, an S-module $D(\mathcal{A})$ is defined by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H \ (\forall H \in \mathcal{A})\}.$$

Roughly speaking, $D(\mathcal{A})$ consists of the vector fields tangent to hyperplanes in $\mathcal{A}$. We want to consider the structure of this module. Here we introduce an example of interesting structures of $D(\mathcal{A})$ determined by the combinatorics. We say $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module. If $\mathcal{A}$ is free with a homogeneous basis $\theta_1, \ldots, \theta_l \in D(\mathcal{A})$, then the *exponents* of $\mathcal{A}$ are defined by $(\deg(\theta_1), \ldots, \deg(\theta_l))$, where $\deg(\theta_i) := \deg(\theta_i(\alpha_i))$ for some linear form α_i with $\theta_i(\alpha_i) \neq 0$. It is easy to see the exponents of a free arrangement do not depend on the choice of bases. For a free arrangement, its combinatorics and the structure of $D(\mathcal{A})$ are related as follows:

Theorem 1.2 (Terao's factorization)
For a free arrangement $\mathcal{A}$ with exponents $(d_1, \ldots, d_l)$, it holds that

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{l} (t - d_i).$$

In particular, this theorem shows the degrees of a basis of a free arrangement depend only on its combinatorics. One of the most important problems related with the logarithmic vector fields is the following conjecture due to Terao, which is on the relation between the freeness and the combinatorics.

Conjecture 1.1 (Terao)
A freeness of an arrangement depends only on its combinatorics.

2 Stability of arrangements and main results

2.1 Main theorem

In this section we assume all arrangements are non-empty and central. Then it is obvious that

$$\theta_E := \sum_{i=1}^{l} X_i \frac{\partial}{\partial X_i} \in D(\mathcal{A}).$$

Define

$$\begin{aligned} D_0(\mathcal{A}) : &= D(\mathcal{A})/S \cdot \theta_E, \text{ and} \\ E(\mathcal{A}) : &= \widetilde{D_0(\mathcal{A})}. \end{aligned}$$

It is known that $E(\mathcal{A})$ is a rank $l-1$ reflexive sheaf on $\mathbf{P}(V) \simeq \mathbf{P}_{\mathbb{K}}^{l-1}$, where we say a torsion free sheaf on the projective space is *reflexive* if $E \simeq E^{**}$. From now on, we assume $l = 3$ and $\mathbb{K}$ is an algebraically closed field of characteristic zero. In this case $E(\mathcal{A})$ is a rank two vector bundle on $\mathbf{P}^2$. Recently instead of $D(\mathcal{A})$ the sheaf $E(\mathcal{A})$ is studied by using algebraic geometry and such studies give a new insight into the arrangement theory. In this section we consider the stability of $E(\mathcal{A})$, where the stability of torsion free sheaves is defined as follows:

Definition 2.1
A torsion free sheaf E on $\mathbf{P}^{l-1}$ is said to be stable if for any subsheaf $F \subset E$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$, it holds that

$$\frac{c_1(F)}{\mathrm{rank}(F)} < \frac{c_1(E)}{\mathrm{rank}(E)}.$$

Moreover, we say an l-arrangement is stable if $E(\mathcal{A})$ is stable on $\mathbf{P}(V)$.

To apply algebraic geometry it is important to consider the stability of $E(\mathcal{A})$, since the stability enables us, for example, to use the Beilinson's monad and so on. However, there are few studies on the stability of arrangements, e.g., those which are normal crossing. We consider the stability of arrangements which are not normal crossing, especially that of a family of rank two vector bundles $E(\mathcal{A}(k))$ for families of 3-arrangements $\{\mathcal{A}(k)\}$ defined as follows:

Definition 2.2
A family of arrangements $\{\mathcal{A}(k)\}_{k\in\mathbb{Z}_{>0}}$ *is called a family of* A_2*-type arrangements if each* $\mathcal{A}(k)$ *is defined as follows:*

$$\begin{aligned} X &= (-k+1)Z,\dots,(k+c-1)Z\ (c\geq 0),\\ X &= (-k+1)Z,\dots,(k+f-1)Z\ (f\geq 0),\\ Y+X &= (-k+a)Z,\dots,(k+a+b-1)Z\ (b\geq -1),\\ Z &= 0, \end{aligned}$$

here $a,b,c,f\in\mathbb{Z}$*.*

When $(a,b,c,f)=(1,-1,0,0)$ and $k=1$ this is the coning of the Coxeter arrangement of type A_2. Hence we call this a family of A_2-type arrangements, and by the same way, we can define arrangements of other Coxeter types, e.g., a family of B_2-type arrangements. The main theorem is the complete classification of families of A_2-type arrangements from the view point of the freeness and stability.

Theorem 2.1
Let $\{\mathcal{A}(k)\}$ *be a family of* A_2*-type arrangements defined in Definition 2.2. By the induction and the proper choice of coordinates, we may assume* $f=0$ *or* 1*. Let us put* $N:=2a+b-c-f$*. Then the following hold:*

(a) For $k\gg 0$*,* $\mathcal{A}(k)$ *is free if and only if* $N=0,\ 1,\ 2$*.*

(b) For $k\gg 0$*,* $\mathcal{A}(k)$ *is stable if and only if* $N>2$ *or* $N<0$*.*

In particular, Theorem 2.1 shows the stability and the freeness of the family of A_2-type arrangements are determined by the combinatorics (more precisely, by the characteristic polynomials). We can show the similar results for the family of B_2-type arrangements, and these pose the problem whether the stability of arrangements (especially those of 3-arrangements) are determined by the combinatorics or not. As an application of Theorem 2.1, we give a partial answer to some problem on the relation between the combinatorics and geometry of A_2-type arrangements.

References

[OT] P. Orlik and H. Terao, Arrangements of hyperplanes. Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.