

Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system

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1 Introduction

We consider the Cauchy problem for the following hyperbolic-elliptic system:

$$u_t + (u^2/2)_x + q_x = 0, \quad (1.1)$$

$$\partial_x^4 q - \partial_x^2 q + q + u_x = 0, \quad (1.2)$$

$$u(0, x) = u_0(x), \quad (1.3)$$

where $u = u(t, x)$ and $q = q(t, x)$ are unknown functions of $t > 0$ and $x \in \mathbb{R}$.

This system has a dissipative structure described by $\lambda(i\xi) = -\frac{\xi^2}{1+\xi^2+\xi^4}$, where $\lambda(i\xi)$ denotes the eigenvalue of the corresponding linearized system. A similar dissipative structure which is characterized by

$$\operatorname{Re}\lambda(i\xi) \leq -\frac{c\xi^2}{(1+\xi^2)^2} \quad (1.4)$$

was observed in [3] and [1] for the dissipative Timoshenko system.

The main purpose is to prove the global existence and asymptotic decay of solutions to the Cauchy problem (1.1), (1.2), (1.3). For our system (1.1), (1.2) with the dissipative structure characterized by (1.4), we will observe that regularity-loss occurs not only in the dissipative part of the usual energy estimates but also in the decay estimates for the linearized system. Such a regularity-loss property causes a serious difficulty in showing the global a priori estimates of solutions to the nonlinear problem. To resolve this difficulty, we introduce a time-weighted energy method. This idea combined with the optimal decay estimates for lower order derivatives of solutions yields the desired global a priori estimates for the problem (1.1), (1.2), (1.3). We also prove that the global solution is asymptotic to the self-similar solution to the Burgers equation as time tends to infinity.

2 Main Theorems

Our first theorem is concerning the global existence and optimal decay of solutions to the Cauchy problem (1.1), (1.2), (1.3).

Theorem 2.1 *Let $s \geq 7$. Assume that $u_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ and put $E_0 = \|u_0\|_{H^s} + \|u_0\|_{L^1}$. Then there is a small positive constant δ_0 such that if $E_0 \leq \delta_0$, then the Cauchy problem (1.1), (1.2), (1.3) has a unique global solution $(u, q)(t, x)$ with*

$$u \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R})), \quad q \in C([0, \infty); H^{s+3}(\mathbb{R})).$$

The solution verifies the following optimal decay estimates:

$$\|\partial_x^k u(t)\|_{L^2} \leq CE_0(1+t)^{-\frac{1}{4}-\frac{k}{2}} \quad (2.1)$$

for k with $0 \leq k \leq \left[\frac{s-1}{2}\right] - 1$ and

$$\|\partial_x^k q(t)\|_{H^4} \leq CE_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad (2.2)$$

for k with $0 \leq k \leq \left[\frac{s-1}{2}\right] - 2$.

Nonlinear diffusion wave

It is well known that $\phi(t, x; M)$ which is the self-similar solution to the Burgers equation $\phi_t + (\phi^2/2)_x = \phi_{xx}$ with the integral condition $\int_{\mathbb{R}} \phi(t, x; M) dx = M$ is given explicitly as

$$\phi(t, x; M) = \frac{1}{\sqrt{t}} \Phi\left(\frac{x}{\sqrt{t}}; M\right) := \frac{1}{\sqrt{t}} \frac{(e^{\frac{M}{2}} - 1)e^{-\xi^2}}{\sqrt{\pi} + (e^{\frac{M}{2}} - 1) \int_{\xi}^{\infty} e^{-\eta^2} d\eta}, \quad (2.3)$$

where $\xi = \frac{x}{\sqrt{4t}}$. Note that $v(t, x) := \phi(t+1, x; M) = \frac{1}{\sqrt{t+1}} \Phi\left(\frac{x}{\sqrt{t+1}}; M\right)$ becomes a solution to the Cauchy problem for the Burgers equation:

$$\begin{cases} v_t + (v^2/2)_x = v_{xx}, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ v(0, x) = v_0(x) := \Phi(x), & x \in \mathbb{R}. \end{cases} \quad (2.4)$$

Our result on the asymptotic self-similar profile of the global solution constructed in Theorem 2.1 is then stated as follows.

Theorem 2.2 *Let $s \geq 7$. Assume that $u_0 \in H^s(\mathbb{R}) \cap L^1_1(\mathbb{R})$ and put $E_1 = \|u_0\|_{H^s} + \|u_0\|_{L^1_1}$. Let $(u, q)(t, x)$ be the global solution to the problem (1.1), (1.2), (1.3) which was constructed in Theorem 2.1, and let $v(t, x) = \phi(t+1, x; M)$ be the self-similar solution to the Burgers equation that was given above with $M = \int_{\mathbb{R}} u_0(x) dx$. Then for any $\epsilon > 0$, there is a small positive constant δ_1 such that if $E_1 \leq \delta_1$, then we have the following asymptotic relations:*

$$\|\partial_x^k(u - v)(t)\|_{L^2} \leq CE_1(1+t)^{-\frac{3}{4} - \frac{k}{2} + \epsilon} \quad (2.5)$$

for k with $0 \leq k \leq \lfloor \frac{s-1}{2} \rfloor - 2$ and

$$\|\partial_x^k(q + v_x)(t)\|_{H^4} \leq CE_1(1+t)^{-\frac{5}{4} - \frac{k}{2} + \epsilon} \quad (2.6)$$

for k with $0 \leq k \leq \lfloor \frac{s-1}{2} \rfloor - 3$.

3 Decay estimates for linearized system

In this section, we study decay property of solutions to the linearized system

$$u_t + q_x = 0, \quad (3.7)$$

$$\partial_x^4 q - \partial_x^2 q + q + u_x = 0, \quad (3.8)$$

with the initial condition $u(0, x) = u_0(x)$. By taking the Fourier transform and eliminating \hat{q} , we arrive at the the expression $\hat{u}(t, \xi) = e^{-\rho(\xi)t} \hat{u}_0(\xi)$, where $\rho(\xi) = \frac{\xi^2}{1+\xi^2+\xi^4}$. We define the semigroup e^{tA} associated with the linearized system (3.7), (3.8) by

$$u(t) = e^{tA} u_0 := \mathcal{F}^{-1} e^{-\rho(\xi)t} \mathcal{F} u_0. \quad (3.9)$$

We also introduce the semigroup e^{tA_0} associated with the linear heat equation $u_t = u_{xx}$:

$$e^{tA_0} u_0 := \mathcal{F}^{-1} e^{-\xi^2 t} \mathcal{F} u_0. \quad (3.10)$$

Note that the Duhamel principle implies that the solution $u(t)$ of the nonlinear system (1.1), (1.2) solves the integral equation

$$u(t) = e^{tA} u_0 - \int_0^t e^{(t-\tau)A} (u^2/2)_x(\tau) d\tau. \quad (3.11)$$

Now we derive qualitative decay estimates for the semigroup e^{tA} .

Lemma 3.1 *Let e^{tA} be the semigroup for the linearized system (3.7), (3.8) defined in (3.9). Then we have*

$$\|\partial_x^k e^{tA} \phi\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|\phi\|_{L^1} + C(1+t)^{-\frac{l}{2}} \|\partial_x^{k+l} \phi\|_{L^2} \quad (3.12)$$

for $k, l \geq 0$. e^{tA} can be approximated by the semigroup e^{tA_0} in (3.10) in the following sense:

$$\|\partial_x^k (e^{tA} - e^{tA_0}) \phi\|_{L^2} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}} \|\phi\|_{L^1} + C(1+t)^{-\frac{l}{2}} \|\partial_x^{k+l} \phi\|_{L^2} \quad (3.13)$$

for $k, l \geq 0$.

Remark The second term on the right hand side of (3.12) shows that we can get the qualitative decay rate $t^{-\frac{l}{2}}$ at the consumption of the l -th order regularity on the initial data.

4 Proof of Theorem 2.1

We define a time-weighted energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \sup_{0 \leq \tau \leq t} (1+\tau)^{j-\frac{1}{2}} \|\partial_x^j u(\tau)\|_{H^{s-2j}}^2, \quad D(t)^2 = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \int_0^t (1+\tau)^{j-\frac{3}{2}} \|\partial_x^j u(\tau)\|_{H^{s-2j}}^2 d\tau. \quad (4.1)$$

To obtain the optimal decay rate, we introduce

$$M(t) = \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor - 1} \sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{1}{4}+\frac{j}{2}} \|\partial_x^j u(\tau)\|_{L^2}. \quad (4.2)$$

STEP 1 We show that for $0 \leq j \leq \lfloor \frac{s}{2} \rfloor$,

$$(1+t)^{j-\frac{1}{2}} \|\partial_x^j u(t)\|_{H^{s-2j}}^2 + \int_0^t (1+\tau)^{j-\frac{3}{2}} \|\partial_x^j u(\tau)\|_{H^{s-2j}}^2 d\tau \leq C \|u_0\|_{H^s}^2 + CM(t)D(t)^2. \quad (4.3)$$

We apply ∂_x^k to (1.1) and (1.2) and multiply them by $\partial_x^k u$ and $\partial_x^k q$, respectively. After integrating with respect to x , we arrive at, using also the Gagliardo-Nirenberg type inequality,

$$\frac{d}{dt} \|\partial_x^k u\|_{L^2}^2 + 2 \|\partial_x^k q\|_{H^2}^2 \leq C \|u_x\|_{L^\infty} \|\partial_x^k u\|_{L^2}^2, \quad (4.4)$$

where $k \geq 0$. By multiplying (4.4) by $(1+t)^\alpha$ and integrating with respect to t , we get

$$\begin{aligned} & (1+t)^\alpha \|\partial_x^k u(t)\|_{L^2}^2 + 2 \int_0^t (1+\tau)^\alpha \|\partial_x^k q(\tau)\|_{H^2}^2 d\tau \\ & \leq \|\partial_x^k u_0\|_{L^2}^2 + \alpha \int_0^t (1+\tau)^{\alpha-1} \|\partial_x^k u(\tau)\|_{L^2}^2 d\tau + C \int_0^t (1+\tau)^\alpha \|u_x(\tau)\|_{L^\infty} \|\partial_x^k u(\tau)\|_{L^2}^2 d\tau, \end{aligned} \quad (4.5)$$

where $k \geq 0$ and $\alpha \in \mathbb{R}$. Now we take $\alpha = -\frac{1}{2}$ (negative) in (4.5) and add for k with $0 \leq k \leq s$. This gives

$$\begin{aligned} & (1+t)^{-\frac{1}{2}} \|u(t)\|_{H^s}^2 + 2 \int_0^t (1+\tau)^{-\frac{1}{2}} \|q(\tau)\|_{H^{s+2}}^2 d\tau + \frac{1}{2} \int_0^t (1+\tau)^{-\frac{3}{2}} \|u(\tau)\|_{H^s}^2 d\tau \\ & \leq \|u_0\|_{H^s}^2 + C \int_0^t (1+\tau)^{-\frac{1}{2}} \|u_x(\tau)\|_{L^\infty} \|u(\tau)\|_{H^s}^2 d\tau, \end{aligned} \quad (4.6)$$

where the third term on the right hand side is an artificial dissipation term which was introduced by choosing $\alpha < 0$. Here the last term on the right hand side can be majorized by $CM(t)D(t)^2$. Thus we have shown (4.3) for $j = 0$. We have from (1.2) that $\|\partial_x u\|_{L^2} \leq C\|q\|_{H^4}$, which together with (4.6) implies

$$\int_0^t (1+\tau)^{-\frac{1}{2}} \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \leq C\|u_0\|_{H^s}^2 + CM(t)D(t)^2. \quad (4.7)$$

Next, we take $\alpha = \frac{1}{2}$ in (4.5) and add for k with $1 \leq k \leq s-1$. This gives

$$\begin{aligned} & (1+t)^{\frac{1}{2}} \|\partial_x u(t)\|_{H^{s-2}}^2 + 2 \int_0^t (1+\tau)^{\frac{1}{2}} \|\partial_x q(\tau)\|_{H^s}^2 d\tau \\ & \leq \|\partial_x u_0\|_{H^{s-2}}^2 + C \int_0^t (1+\tau)^{-\frac{1}{2}} \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau + C \int_0^t (1+\tau)^{\frac{1}{2}} \|u_x(\tau)\|_{L^\infty} \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau. \end{aligned} \quad (4.8)$$

Here the second term of the right hand side of (4.8) was estimated in (4.7), while the third term can be majorized by $CM(t)D(t)^2$. Thus we have proved (4.3) for $j = 1$. The general case can be shown by using induction argument.

STEP 2 Next we show that

$$M(t) \leq C(\|u_0\|_{H^s} + \|u_0\|_{L^1}) + CM(t)^2 + CM(t)E(t). \quad (4.9)$$

Let $0 \leq k \leq [\frac{s-1}{2}] - 1$. We apply ∂_x^k to the integral equation (3.11) and take the L^2 norm. We have

$$\begin{aligned} \|\partial_x^k u(t)\|_{L^2} & \leq \|\partial_x^k e^{tA} u_0\|_{L^2} + \frac{1}{2} \int_0^{\frac{t}{2}} \|\partial_x^{k+1} e^{(t-\tau)A} (u^2)(\tau)\|_{L^2} d\tau + \\ & + \frac{1}{2} \int_{\frac{t}{2}}^t \|\partial_x e^{(t-\tau)A} \partial_x^k (u^2)(\tau)\|_{L^2} d\tau =: I_1 + I_2 + I_3. \end{aligned} \quad (4.10)$$

Here we only give the estimate for I_2 . By applying (3.12) with k replaced by $k+1$ and with $l = k+1$, $\phi = u^2$, we have

$$\begin{aligned} I_2 & \leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{4}-\frac{k+1}{2}} \|(u^2)(\tau)\|_{L^1} d\tau + \\ & + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{k+1}{2}} \|\partial_x^{2k+2} (u^2)(\tau)\|_{L^2} d\tau \\ & \leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \|u(\tau)\|_{L^2}^2 d\tau + \\ & + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{2}-\frac{k}{2}} \|u(\tau)\|_{L^\infty} \|\partial_x^{2k+2} u(\tau)\|_{L^2} d\tau. \end{aligned} \quad (4.11)$$

Now, recalling the definitions of $M(t)$, we have $\|u(t)\|_{L^2} \leq M(t)(1+t)^{-\frac{1}{4}}$ and $\|u(t)\|_{L^\infty} \leq M(t)(1+t)^{-\frac{1}{2}}$. Also, we have from (4.1) that $\|\partial_x^{2k+2} u(t)\|_{L^2} \leq E(t)(1+t)^{-\frac{1}{4}}$ because $2k+2 \leq$

$s - 1$ for $k \leq \lceil \frac{s-1}{2} \rceil - 1$. Substituting these inequalities into (4.11), we can further estimate I_2 as

$$\begin{aligned} I_2 &\leq CM(t)^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-\frac{1}{2}} d\tau + \\ &\quad + CM(t)E(t) \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{2}-\frac{k}{2}} (1+\tau)^{-\frac{3}{4}} d\tau \\ &\leq CM(t)^2 (1+t)^{-\frac{1}{4}-\frac{k}{2}} + CM(t)E(t)(1+t)^{-\frac{1}{4}-\frac{k}{2}}. \end{aligned}$$

The other terms I_1 and I_2 can be estimated similarly. Consequently, we arrive at the inequality

$$(1+t)^{\frac{1}{4}+\frac{k}{2}} \|\partial_x^k u(t)\|_{L^2} \leq C(\|u_0\|_{H^s} + \|u_0\|_{L^1}) + CM(t)^2 + CM(t)E(t)$$

for $0 \leq k \leq \lceil \frac{s-1}{2} \rceil - 1$, which shows the desired estimate (4.9).

Proof of Theorem 2.1. We have from (4.3) and (4.9) that

$$\begin{aligned} E(t)^2 + D(t)^2 &\leq CE_0^2 + CM(t)D(t)^2, \\ M(t) &\leq CE_0 + CM(t)^2 + CM(t)E(t), \end{aligned}$$

where $E_0 = \|u_0\|_{H^s} + \|u_0\|_{L^1}$. Thus, setting $Y(t) := E(t) + D(t) + M(t)$, we arrive at the inequality $Y(t)^2 \leq CE_0^2 + CY(t)^3 + CY(t)^4$, from which we can deduce that $Y(t) \leq CE_0$, provided that E_0 is suitably small, say $E_0 \leq \delta_0$. This gives the desired a priori estimates of solutions, by which we can continue a unique local solution globally in time. The global solution thus obtained verifies the decay estimate (2.1) for $0 \leq k \leq \lceil \frac{s-1}{2} \rceil - 1$ because we have shown that $M(t) \leq CE_0$. The remaining decay estimate (2.2) for q easily follows from (1.2) and (2.1). In fact, we have

$$\|\partial_x^k q(t)\|_{H^4} \leq C\|\partial_x^{k+1} u(t)\|_{L^2} \leq CE_0(1+t)^{-\frac{3}{4}-\frac{k}{2}},$$

where $0 \leq k \leq \lceil \frac{s-1}{2} \rceil - 2$. Thus the proof of Theorem 2.1 is complete. \square

References

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