

A sharp bilinear estimates related with the Schrödinger-improved Boussinesq system

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We consider the Schrödinger-improved Boussinesq system:

$$\begin{cases} (i\partial_t + \Delta)u = vu, \\ (\partial_t^2 - \Delta - \Delta\partial_t^2)v = \Delta|u|^2, \end{cases} \quad (1)$$

where u and v are complex and real valued on $\mathbb{R} \times \mathbb{R}^n$, respectively.

The system (1) is a substitute for the Zakharov system (cf. [2] and references therein). We prescribe the data (u_0, v_0, v_1) for $(u, v, \partial_t v)$ at the initial time $t = 0$. We are interested in the existence of the solution to (1).

Our main result is the following.

Theorem 1 *Let $n \leq 3$. If $s_1 > -1/4$ and $s_2 > -1/2$, then (1) has a solution for data $(u_0, v_0, v_1) \in H^{s_1}(\mathbb{R}^n) \times H^{s_2}(\mathbb{R}^n) \times H^{s_2}(\mathbb{R}^n)$.*

Recently, Ozawa and Tsutaya [2] studied the Cauchy problem for (1) and proved the existence of the unique solution for the data $(u_0, v_0, v_1) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for $n \leq 3$. Thus our theorem is an improvement of their result.

The proof of Theorem 1 is based on the argument of Bourgain [3]. To illustrate the argument of Bourgain, we consider the following abstract nonlinear equation

$$(i\partial_t - H)u(t, x) = u^2(t, x), \quad (2)$$

where $u = u(t, x): \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ and H is a linear operator in the class of tempered distributions on \mathbb{R}^n . In the case where $H = \Delta = \sum_{j=1}^n \partial_j^2$, (2) is a nonlinear Schrödinger equation, where ∂_j denotes the partial differential operator with respect to the j -th component of x .

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We are interested in the existence of the solution to (2). We prescribe a datum f for u at the initial time $t = 0$. We formally have the integral equation associated to (2):

$$u(t) = e^{itH} f - i \int_0^t e^{i(t-t')H} u^2(t') dt', \quad (3)$$

where e^{itH} is the solution map of the corresponding linear equation $(i\partial_t - H)u = 0$.

Now we introduce the several notations. We use $C(\cdot, \dots, \cdot)$ to denote various constants depending on the quantities in the parentheses, which may vary from line to line. We use \mathcal{F}_t and \mathcal{F}_x to denote the Fourier transform in t and x , respectively. Moreover, we set $\mathcal{F} = \mathcal{F}_t \mathcal{F}_x$. For a real valued function h on \mathbb{R}^n , we use $h(\nabla)$ to denote the operator defined by $\mathcal{F}_x[h(\nabla)u] = h\mathcal{F}_x[u]$, thus we have $\Delta = -|\nabla|^2$. Also we use $w(\partial_t)$ to denote the operator defined by $\mathcal{F}_t[w(\partial_t)u] = w\mathcal{F}_t[u]$ for a real valued function w on \mathbb{R} . We write the convolution of the functions F and G on $\mathbb{R} \times \mathbb{R}^n$ as follows:

$$\int_{\substack{\tau=\tau_1+\tau_2 \\ \xi=\xi_1+\xi_2}} F(\tau_1, \xi_1) G(\tau_2, \xi_2) d\tau_1 d\xi_1.$$

We consider (2) in a special case $H = h(\nabla)$ for a real valued function h on \mathbb{R}^n . Then we introduce the quantity

$$\|u\|_{X_{h(\nabla)}^{s,b}} = \|\langle \nabla \rangle^s \langle \partial_t \rangle^b e^{-ith(\nabla)} u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}. \quad (4)$$

We note that, by the Plancherel theorem, (4) is rewritten in the form

$$\|u\|_{X_{h(\nabla)}^{s,b}} = \|\langle \xi \rangle^s \langle \tau + h(\xi) \rangle^b \mathcal{F}u\|_{L^2(\mathbb{R}_\tau \times \mathbb{R}_\xi^n)}. \quad (5)$$

We define the Bourgain space $X_{h(\nabla)}^{s,b}$ as the completion of the space of Schwartz functions by $\|\cdot\|_{X_{h(\nabla)}^{s,b}}$. For an interval I in \mathbb{R} , we define

$$X_{h(\nabla)}^{s,b}(I) = \{u : I \times \mathbb{R}^n \rightarrow \mathbb{C} : \exists \tilde{u} \in X_{h(\nabla)}^{s,b} \text{ s.t. } u = \tilde{u}|_I\}$$

which is equipped with the norm

$$\|u\|_{X_{h(\nabla)}^{s,b}(I)} = \inf\{\|\tilde{u}\|_{X_{h(\nabla)}^{s,b}} : \tilde{u} \in X_{h(\nabla)}^{s,b} \text{ s.t. } u = \tilde{u}|_I\}.$$

We seek for a (time-local) solution u of (2) with $H = h(\nabla)$ by the fixed point theorem. For this, we define the map S by

$$S(u)(t) = e^{ith(\nabla)} f - i \int_0^t e^{i(t-t')h(\nabla)} u^2(t') dt'.$$

Clearly the fixed point $u = S(u)$ is a solution of (2). Note that $u \in X_{h(\nabla)}^{s,b}([-T, T])$ implies that $u \in C([-T, T], H^s)$, if $b > 1/2$.

Let $B_R(T)$ be the ball in $X_{h(\nabla)}^{s,b}([-T, T])$ with radius R . We first show that S maps $B_R(T)$ into itself for sufficiently large R and sufficiently small T . What we have to show is $\|S(u)\|_{X_{h(\nabla)}^{s,b}([-T, T])} \leq R$ for all $u \in B_R(T)$. Assume that $0 < T < 1$. Let ρ be a smooth cut off function on \mathbb{R} such that $\rho = 1$ on $[-1, 1]$ and $\rho = 0$ outside of $[-2, 2]$. We set $\rho_T(t) = \rho(t/T)$. Take any $\tilde{u} \in X_{h(\nabla)}^{s,b}$ such that $\tilde{u} = u$ on $[-T, T]$. Then we have

$$\|S(u)\|_{X_{h(\nabla)}^{s,b}([-T, T])} \leq \|\rho e^{ith(\nabla)} f\|_{X_{h(\nabla)}^{s,b}} + \|\rho_T \int_0^t e^{i(t-t')h(\nabla)} \tilde{u}^2(t') dt'\|_{X_{h(\nabla)}^{s,b}}. \quad (6)$$

By the definition (4), the first term on the R.H.S. of (6) is equal to

$$\|\langle \partial_t \rangle^b \rho \langle \nabla \rangle^s f\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} = \|\rho\|_{H^b(\mathbb{R})} \|f\|_{H^s(\mathbb{R}^n)}. \quad (7)$$

On the other hand, the second term on the R.H.S. of (6) is bounded by (cf. [1])

$$C(\rho, b) \|\tilde{u}^2\|_{X_{h(\nabla)}^{s,b-1}}. \quad (8)$$

By (5) and the duality, the factor $\|\tilde{u}^2\|_{X_{h(\nabla)}^{s,b-1}}$ in (8) is rewritten in the form

$$\sup_{\substack{F \in L^2(\mathbb{R} \times \mathbb{R}^n) \\ \|F\|_{L^2} = 1}} \left| \int d\tau d\xi \int_{\substack{\tau = \tau_1 + \tau_2 \\ \xi = \xi_1 + \xi_2}} d\tau_1 d\xi_1 \frac{\langle \xi \rangle^s}{\langle \tau + \xi \rangle^{1-b}} F(\tau, \xi) \mathcal{F}[\tilde{u}](\tau_1, \xi_1) \mathcal{F}[\tilde{u}](\tau_2, \xi_2) \right|. \quad (9)$$

We set $G = \langle \xi \rangle^s \langle \tau + h(\xi) \rangle^b \mathcal{F}[\tilde{u}]$. Note that $\|G\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} = \|\tilde{u}\|_{X_{h(\nabla)}^{s,b-1}}$. Then (9) is bounded by

$$\sup_{\substack{F \in L^2(\mathbb{R} \times \mathbb{R}^n) \\ \|F\|_{L^2} = 1}} \int d\tau d\xi \int_{\substack{\tau = \tau_1 + \tau_2 \\ \xi = \xi_1 + \xi_2}} d\tau_1 d\xi_1 \frac{\langle \xi \rangle^s \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} |F(\tau, \xi)| |G(\tau_1, \xi_1)| |G(\tau_2, \xi_2)|}{\langle \tau + h(\xi) \rangle^{1-b} \langle \tau_1 + h(\xi_1) \rangle^b \langle \tau_2 + h(\xi_2) \rangle^b}. \quad (10)$$

The main part of the argument is to bound (10) by

$$T^\theta \|F\|_{L^2} \|G\|_{L^2}^2 = T^\theta \|\tilde{u}\|_{X_{h(\nabla)}^{s,b-1}}^2$$

for some $\theta > 0$. Then, since \tilde{u} is arbitrary, we obtain the bound

$$T^\theta C(\rho, b) \|u\|_{X_{h(\nabla)}^{s,b-1}([-T, T])}^2 \leq T^\theta C(\rho, b) R^2$$

for the term (8). Combining this bound and (7), we have

$$\|S(u)\|_{X_{h(\nabla)}^{s,b}([-T, T])} \leq \|\rho\|_{H^b} \|f\|_{H^s} + C(\rho, b) T^\theta R^2.$$

Finally, we choose $R = 2\|\rho\|_{H^b}\|f\|_{H^s}$ and take T so small that $T^\theta C(\rho, b)R^2 \leq R/2$. Then we have $\|S(u)\|_{X_{h(\nabla)}^{s,b}([-T, T])} \leq R$. Thus S is the map in $X_{h(\nabla)}^{s,b}([-T, T])$.

By a way similar to the above, we show that S is a contraction map in $B_R(T)$, if we take T sufficiently small. Thus we see the existence of the solution.

We comment on the estimate for (10). We set $\lambda = \tau + h(\xi)$, $\lambda_1 = \tau_1 + h(\xi_1)$ and $\lambda_2 = \tau_2 + h(\xi_2)$ in (10). We easily see that

$$\begin{aligned} |h(\xi) - h(\xi_1) - h(\xi_2)| &= |\tau - \tau_1 - \tau_2 + h(\xi) - h(\xi_1) - h(\xi_2)| \\ &\leq 3 \max\{|\lambda|, |\lambda_1|, |\lambda_2|\}. \end{aligned} \quad (11)$$

for (τ, τ_1, τ_2) with $\tau = \tau_1 + \tau_2$. The relation (11) plays an important role to estimate (10). In particular, in the case where $h(\xi) = \xi^3$ ($\xi \in \mathbb{R}$), we have

$$|\xi||\xi_1||\xi_2| \leq \max\{|\lambda|, |\lambda_1|, |\lambda_2|\}.$$

The key estimates for the proof of Theorem 1 are the following bilinear estimates which correspond to the estimate for (8).

Theorem 2 *Let $n \leq 3$, $s_1, s_2 < 0$ and $0 < b < 1/2$.*

(i) *If $s_1, s_2 > -1/2$, then we have*

$$\|\psi\phi\|_{X_{|\nabla|^2}^{s_1, -b}} \leq C(s_1, s_2, b, n) \|\psi\|_{X_{|\nabla|^2}^{s_1, b}} \|\phi\|_{X_{\omega(\nabla)}^{s_2, b}} \quad (12)$$

for some b sufficiently close to $1/2$, where we may replace ϕ with $\bar{\phi}$ on the L.H.S. of (12).

(ii) *If $s_1 > -1/4$ and $s_2 > -1$, then we have*

$$\|\psi^2\|_{X_{\omega(\nabla)}^{s_1, -b}} \leq C(s_1, s_2, b, n) \|\psi\|_{X_{|\nabla|^2}^{s_1, b}}^2 \quad (13)$$

for some b sufficiently close to $1/2$.

References

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