GOTZMANN MONOMIAL IDEALS

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1. PRELIMINARIES

Let K be an arbitrary field and $R = K[x_1, x_2, \ldots, x_n]$ the polynomial ring in n variables over a field K with each $\deg(x_i) = 1$. Let $I = \bigoplus_{d \geq 0} I_d$ be a homogeneous ideal of R, where each $I_d = \{f \in I : \deg(f) = d\}$ denotes the d-th homogeneous component of I. The function $H(I, -) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ with $H(I, d) = \dim_K(I_d)$ for all $d \in \mathbb{Z}_{\geq 0}$ is *the Hilbert function of I.*

Macaulay proved that, for each $d \geq 0$, there is a lower bound for $H(I, d+1)$ in terms of $H(I, d)$ by using following representations:

Let n and h be positive integers. Then h can be written uniquely in the form, called the *nth binomial representation of h,*

$$h = \binom{h(n) + n}{n} + \binom{h(n-1) + n - 1}{n-1} + \cdots + \binom{h(i) + i}{i},$$

where $h(n) \geq h(n-1) \geq \cdots \geq h(i) \geq 0$, $i \geq 1$.

If $h = \binom{h(n)+n}{n} + \binom{h(n-1)+n-1}{n-1} + \cdots + \binom{h(i)+i}{i}$ is the nth binomial representation of h, then we define

$$h^{<n>} = \binom{h(n) + n + 1}{n} + \binom{h(n-1) + n}{n-1} + \cdots + \binom{h(i) + i + 1}{i}.$$

Theorem (Minimal growth of Hilbert function)**.** *Let I be a homogeneous ideal of $R = K[x_1, x_2, \ldots, x_n]$. Then one has*

$$H(I, d+1) \geq H(I, d)^{<n-1>}.$$

In 1978, Gotzmann [8] proved an interesting theorem, called Gotzmann's persistence theorem, on the Hilbert function of a homogeneous ideal I of the polynomial ring. We recall Gotzmann's persistence Theorem.

Theorem (**Gotzmann's Persistence Theorem [5]**)**.** *Let $R = K[x_1, \ldots, x_n]$ and I a homogeneous ideal of R generated in degree $\leq d$. If $H(I, d+1) = H(I, d)^{<n-1>}$, then $H(I, k+1) = H(I, k)^{<n-1>}$ for all $k \geq d$.*

A monomial ideal $I \subset R$ is called a *Gotzmann monomial ideal* if I is generated in one degree d and if I satisfies $H(I, d+1) = H(I, d)^{<n-1>}$. (Note that Gotzmann's persistence theorem implies $H(I, k+1) = H(I, k)^{<n-1>}$ for all $k \geq d$.) In case of monomial ideal $I \subset R$, each k-th Hilbert function $H(I, k)$ of I is simply the number of monomials of degree k in I. Thus if V is the set of monomials of degree d in I, then I is Gotzmann if and only if $|MV| = |V|^{<n-1>}$, where $|V|$ is the cardinality of

the finite set V. We are interested in the structure of Gotzmann monomial ideals. Instead of discussing an ideal I itself, we consider the set of monomials of degree d in I.

Let $x_1{}^{a_1} x_2{}^{a_2} \ldots x_n{}^{a_n}$ and $x_1{}^{b_1} x_2{}^{b_2} \ldots x_n{}^{b_n}$ be monomials in R. The *lexicographic order* $<_{lex}$ of R is defined by $x_1{}^{a_1} x_2{}^{a_2} \ldots x_n{}^{a_n} <_{lex} x_1{}^{b_1} x_2{}^{b_2} \ldots x_n{}^{b_n}$ if the leftmost nonzero entry of $(b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$ is positive.

Let M denote the set of variables $\{x_1, x_2, \ldots, x_n\}$ and M^d the set of all monomials of degree d in R, where $M^0 = \{1\}$. Let $V \subset M^d$ be a set of monomials of degree d. For a monomial $u \in R$, we write $uV = \{uv : v \in V\}$ and $MV = \{x_i v : v \in V, \ i = 1, 2, \ldots, n\}$. Also, we write $\gcd(V)$ for the greatest common divisor of the monomials belonging to V.

   (i) $V \subset M^d$ is called a *lexsegment set* if for any $u \in V$ and for any monomial $v \in M^d$ with $v >_{lex} u$ one has $v \in V$. For a positive integer $|M^d| \geq a > 0$, let $Lex(n, d, a) \subset M^d$ denote the unique lexsegment set with $|Lex(n, d, a)| = a$.

   (ii) A subset $V \subset M^d$ is called a *Gotzmann set* if $|MV| = |V|^{<n-1>}$.

It is known that every lexsegment set is Gotzmann.

2. Main Theorems

We define $V \sim V'$ if we can obtain V' from V by a permutation of variables. In other words, there exists a permutation π of $\{1, 2, \ldots, n\}$ such that $\pi(V) = V'$, where for the permutation $\pi = (\pi(1), \ldots, \pi(n))$ of $\{1, 2, \ldots, n\}$, we define $\pi(x_1^{a_1} \ldots x_n^{a_n}) = x_{\pi(1)}^{a_1} \ldots x_{\pi(n)}^{a_n}$ and $\pi(V) = \{\pi(u) | u \in V\}$. If $V \subset M^d$ is a Gotzmann set, then, for any monomial $u \in R$, the set uV is also a Gotzmann set. Thus we often assume $\gcd(V) = 1$.

First, we remark that if a positive integer $a > 0$ is given, then a determines the degree d of Gotzmann sets $V \subset M^d$ with $|V| = a$ and with $\gcd(V) = 1$.

Lemma 2.1. *If $V \subset M^d$ is a Gotzmann set with $\gcd(V) = 1$, then one has $\binom{d-1+n-1}{n-1} < |V| \leq \binom{d+n-1}{n-1}$.*

Every lexsegment set is a Gotzmann set. However, a Gotzmann set is not necessarily lexsegment. The main result is finding all integers $a > 0$ such that every Gotzmann set $V \subset M^d$ with $|V| = a$ and with $\gcd(V) = 1$ satisfies $V \sim Lex(n, d, a)$.

Theorem 2.2. *Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring and $a = \sum_{j=p}^{n-1} \binom{a(j)+j}{j}$ the $(n-1)$th binomial representation of a positive integer $a > 0$. Then the following conditions are equivalent:*

   (i) *$a(n-1) = a(n-2) = \cdots = a(p)$;*

   (ii) *If $V \subset M^d$ is a Gotzmann set with $|V| = a$ and $\gcd(V) = 1$, then d is determined by a and $V \sim Lex(n, d, a)$.*

Let $a = \sum_{j=p}^{n-1} \binom{a(j)+j}{j}$ be the $(n-1)$th binomial representation of a positive integer $a > 0$ and V a Gotzmann set with $|V| = a$ and with $\gcd(V) = 1$. By Theorem 2.2, if $a(p) = a(n-1)$ then V must be a lexsegment set by a proper

permutation of variables. A positive integer $a > 0$ is called an nth *lexnumber*, or simply a *lexnumber* if $a(p) = a(n-1)$.

Example 2.3. Here are some lexnumbers for $n = 3, 4, 5$.

$n = 3$: $1, 2, 3, 5, 6, 9, 10, 14, 15, 20, 21, 27, 28, 35, 36, 44, 45, 54, 55, 65, 66,$
 $77, 78, 90, 91, 104, 105, \ldots$
$n = 4$: $1, 2, 3, 4, 7, 9, 10, 16, 19, 20, 30, 34, 35, 50, 55, 56, 77, 83, 84, 112, \ldots,$
$n = 5$: $1, 2, 3, 4, 5, 9, 12, 14, 15, 25, 31, 34, 35, 55, 65, 104, 105, \ldots,$

For fixed d, there are only $\{d(n-1) + 1\}$ lexnumbers, since there are $(n-1)$ lexnumbers between $\binom{t+n-1}{n-1}$ and $\binom{t+n}{n-1}$.

We also consider Gotzmann sets in a few variables. If $n = 1$, then all sets $V \subset M^d$ are Gotzmann sets. If $n = 2$, we can easily show that $V \subset M^d$ is a Gotzmann sets if and only if $V = \emptyset$ or $V = M^d$, when we assume $\gcd(V) = 1$. We consider the case $n = 3$.

Let V be a set of monomials of degree d and let $u = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ a monomial of degree d. We say that a monomial $v = x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$ of degree d is *under u for i* if $b_j \le a_j$ for all $j \ne i$. We call $u \in M^d$ a *fixed empty element of V for i* any monomial which is under u for i does not belongs to V. Note that, if u is a fixed empty element of V for i, then any monomial v which is under u for i is also a fixed empty element of V for i.

Theorem 2.4. *Let $R = K[x_1, x_2, x_3]$ and $V \subset R$ a set of monomials of degree d with $\gcd(V) = 1$. Then, V is a Gotzmann set if and only if any monomial $v \notin V$ of degree d in R is a fixed empty element of V for some i and $|V| > \binom{d-1+n-1}{n-1}$.*

Example 2.5. To understand the meaning of Theorem 2.4, drawing a picture of monomials is useful. In the picture below, all monomials of degree 4 in $K[x_1, x_2, x_3]$ are displayed. The monomial x_1^4 is in the lower left corner, x_3^4 is in the lower right corner, and x_2^4 is at the top. The black dots denote monomials in V and the empty circles denote monomials which are missing. For example, figure (1) means x_1^4, $x_1^3 x_2$, $x_1^2 x_2^2$ and $x_1 x_2^3$ are missing. In the picture below, we classify all Gotzmann sets V in $K[x_1, x_2, x_3]$ with $\gcd(V) = 1$ and $|V| = \binom{4+2}{2} - 4 = 11$ up to permutations.

(5) (6) (7) (8)

(9) (10)
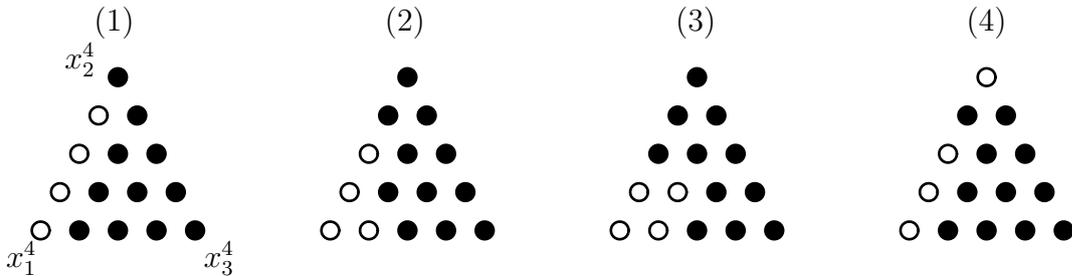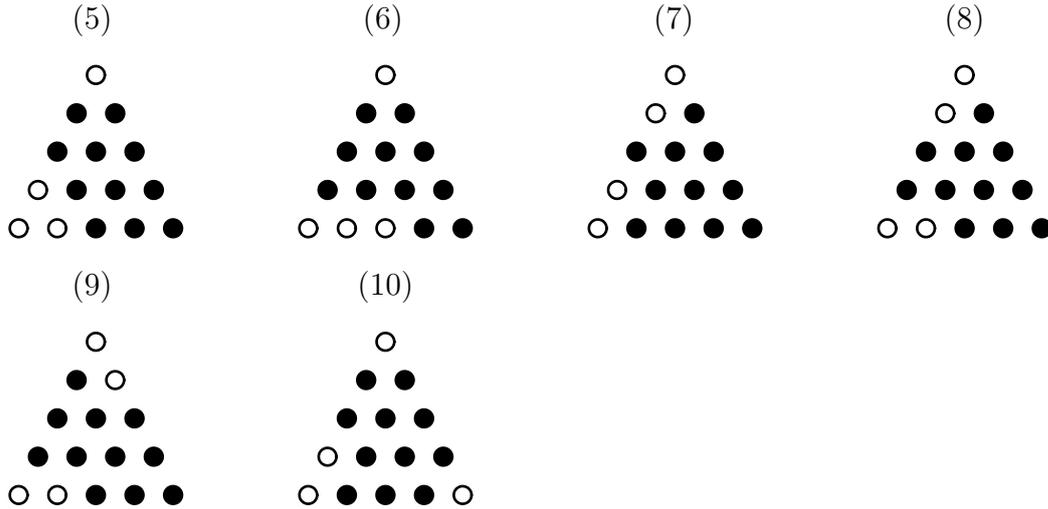
Theorem 2.4 says that each connected component of empty circles must be at the corner. Also, the numbers of empty circles must be equal to or less than the degree of elements of V.

3. Related works

There has been a lot of interest in the extremal properties of Hilbert functions. We note some related works.

Aramova, Herzog and Hibi [2] consider Gotzmann's persistence theorem for the exterior algebra. Furthermore, Gasharov [6] generalized the persistence theorem to finitely generated modules over the polynomial ring and to exterior algebras. It would be interesting to extend the persistence theorem for more general objects. For example, in [7], persistence theorems for the ring $K[x_1, x_2, \ldots, x_n]/(x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$ is considered, where $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \infty$ and where $x_i^\infty = 0$.

Let $I \subset R$ be a homogeneous ideal and let $I_{(d)}$ the ideal generated by all polynomials of degree d in I. A homogeneous ideal I is called Gotzmann if $H(I_{(d)}, d+1) = H(I_{(d)}, d)^{<n-1>}$ for all $d \geq 0$. A monomial ideal $I \subset R$ is called lexsegment if each $\{u \in M^d : u \in I_d\}$ is a lexsegment set. Herzog and Hibi [10] gives an interesting characterization of Gotzmann ideals from the viewpoint of graded Betti numbers. This work is related to the extremal property of lexsegment ideals ([1], [3] and [9]). It would be interesting to consider further properties of Gotzmann ideals and lexsegment ideals.

Results related to Theorem 2.2 have been obtained by Furedi and Griggs [5]. They determined all integers $a > 0$ such that every squarefree Gotzmann set V with $|V| = a$ is unique up to the permutation of variables. It is a challenging problem to determine all Gotzmann sets. However, structures of Gotzmann sets $V \subset K[x_1, x_2, \ldots, x_n]$ are more complicated for $n \geq 4$.

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