

# INTRODUCTION TO THE BORDISM PRINCIPLE

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## 1. WHITNEY THEOREM

A number of problems of differential geometry are known to have counterparts in homotopy theory. If a differential geometry problem is, in fact, equivalent to its homotopy theoretic counterpart, then we say that it satisfies *the homotopy principle*.

Let us, for example, consider *immersions* of a closed curve into the two dimensional plane  $\mathbb{R}^2$ . By definition, an immersion  $f$  of a closed curve into  $\mathbb{R}^2$  is a smooth map of a unit circle  $S^1$  into  $\mathbb{R}^2$  without singularities. To reformulate the definition in terms of coordinates, let  $\varphi$  be the angular coordinate on the circle and  $x, y$  be the standard coordinates on  $\mathbb{R}^2$ . Then the map  $f$  from  $S^1$  into  $\mathbb{R}^2$  can be given by a pair of functions

$$(1) \quad x = x(\varphi),$$
$$(2) \quad y = y(\varphi).$$

The map  $f$  is *an immersion* if the functions (1) and (2) are smooth and for each angle  $\varphi$  at least one of the numbers  $x_\varphi(\varphi) = \frac{dx}{d\varphi}(\varphi)$  or  $y_\varphi(\varphi) = \frac{dy}{d\varphi}(\varphi)$  is not zero.

We say that two immersions  $f$  and  $g$  are *regularly homotopic* if there is a smooth deformation of  $f$  to  $g$  through immersions. More precisely, there is a smooth family of maps  $F_t : S^1 \rightarrow \mathbb{R}^2$  parameterized by a (time) parameter  $t \in [0, 1]$  such that the initial map  $F_0$  coincides with  $f$ , the terminal map  $F_1$  coincides with  $g$ , and at each moment  $t \in [0, 1]$  the map  $F_t$  is an immersion.

One may wonder if any two immersions are regularly homotopic. To answer this question, Whitney proposed to shift from this differential geometry problem to its homotopy theoretic counterpart. Namely, given an immersion  $f(\varphi) = (x(\varphi), y(\varphi))$  of a closed curve  $S^1$ , at each moment  $\varphi_0$  the direction of the velocity vector  $(x_\varphi(\varphi_0), y_\varphi(\varphi_0))$  determines a direction

$$(3) \quad \frac{(x_\varphi(\varphi_0), y_\varphi(\varphi_0))}{\sqrt{x_\varphi^2(\varphi_0) + y_\varphi^2(\varphi_0)}} \in S^1$$

Thus, the immersion  $f$  gives rise to a continuous map

$$G_f : S^1 \rightarrow S^1.$$

that assigns to a moment  $\varphi_0$  the direction (3). Under regular homotopy of  $f$ , the continuous map  $G_f$  deforms continuously, i.e., by homotopy. Consequently, if two immersions  $f, g$  are regularly homotopic, then the maps  $G_f$  and  $G_g$  are homotopic. The converse statement turns out to be also true.

**Theorem 1.1.** (*parametric h-principle for immersions  $S^1 \rightarrow \mathbb{R}^2$* ) *Two immersions  $f$  and  $g$  are regularly homotopic if and only if the maps  $G_f$  and  $G_g$  are homotopic.*

The homotopy theoretic part of Theorem 1.1 is simple; two maps  $G_f$  and  $G_g$  are homotopic if and only if their *degrees* are the same, i.e., the maps  $G_f, G_g : S^1 \rightarrow S^1$  wind the circle  $S^1$  around itself the same number of times.

For example, if  $f$  is the standard embedding of  $S^1$  into  $\mathbb{R}^2$  and  $g$  is the figure “8” immersion, then the degree of  $G_f$  is 1, whereas the degree of  $G_g$  is 0. Hence  $G_f$  is not homotopic to  $G_g$  and therefore  $f$  is not regularly homotopic to  $g$ .

## 2. H-PRINCIPLE FOR IMMERSIONS. SMALE PARADOX

For a manifold  $M$ , let  $\pi_M : TM \rightarrow M$  denote the projection of the tangent bundle  $TM$  of the manifold  $M$ , and  $T_xM$  be the fiber of  $TM$  over a point  $x \in M$ . We say that a smooth map  $f : M \rightarrow N$  of compact manifolds is an *immersion*, if the differential  $df : TM \rightarrow TN$  of  $f$  is a monomorphism of vector bundles, i.e., for each point  $x \in M$ , the differential  $d_x f : T_xM \rightarrow T_xN$  of  $f$  at  $x$  is a monomorphism.

Our next example is the differential geometry problem that in 60-70s inspired a number of further results culminating in the birth of the h-principle theory.

**Problem 1.** *Given a continuous map  $f : M \rightarrow N$  of smooth manifolds. Does there exist an immersion  $M \looparrowright N$  homotopic to the map  $f$ ?*

From the definition of immersion, it follows that a necessary condition for the existence of an immersion homotopic to  $f$  is the existence of a vector bundle monomorphism  $TM \rightarrow TN$  covering the map  $f$ , i.e., the existence of a continuous map  $\tau : TM \rightarrow TN$  such that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\tau} & TN \\ \pi_M \downarrow & & \pi_N \downarrow \\ M & \xrightarrow{f} & N, \end{array}$$

commutes and for each point  $x \in M$ , the map  $\tau|_{T_xM} : T_xM \rightarrow T_yN$ , with  $y = f(x)$ , is a monomorphism of vector spaces. Hence the homotopy theoretic counterpart of Problem 1 is Problem 2.

**Problem 2.** *Given a continuous map  $f : M \rightarrow N$  of smooth manifolds, does there exist a monomorphism  $TM \rightarrow TN$  of tangent bundles covering the map  $f$ .*

The h-principle for immersions, proved by Smale and Hirsch, asserts that Problem 1 and Problem 2 are, in fact, equivalent provided that either the dimension of the manifold  $M$  is strictly less than the dimension of the manifold  $N$ , or the dimensions of  $M$  and  $N$  are the same and the manifold  $M$  is open. Furthermore, two immersions  $f_1, f_2 : M \looparrowright N$  are *regularly homotopic*, i.e., homotopic through immersions, if and only if  $df_1$  and  $df_2$  are homotopic through monomorphisms of tangent bundles.

A surprising consequence of the h-principle for immersions is the famous Smale Paradox.

**Corollary 2.1** (Smale Paradox). *Every immersion  $S^2 \looparrowright \mathbb{R}^3$  is regularly homotopic to the standard embedding. In particular, the standard embedding and its eversion are regularly homotopic.*

*Proof.* Regular homotopy classes of immersions  $f : S^2 \looparrowright \mathbb{R}^3$  are in bijective correspondence with regular homotopy classes of immersions  $\tilde{f} : V \looparrowright \mathbb{R}^3$  of the thickened sphere  $V := S^2 \times [-1, 1]$ . By the h-principle, the latter classes are in bijective correspondence with homotopy classes of monomorphisms

$$TV = V \times \mathbb{R}^3 \longrightarrow T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3,$$

which, in their turn, are in bijective correspondence with homotopy classes of maps  $V \rightarrow \mathbb{R}^3 \times SO_3$ . Since  $\pi_2(SO_3) = 0$ , every map  $V \rightarrow \mathbb{R}^3 \times SO_3$  is homotopic to a constant map. Thus, there is only one regular homotopy class of immersions  $S^2 \looparrowright \mathbb{R}^3$ .  $\square$

### 3. H-PRINCIPLE FOR SUBMERSIONS

If the dimension of a manifold  $M$  is at least as big as the dimension of the manifold  $N$ , then a smooth map  $f : M \rightarrow N$  is called submersion if  $df$  is a nonsingular morphism of tangent bundles, i.e., for each point  $x \in M$ , the homomorphism  $d_x f : T_x M \rightarrow T_y N$ , with  $y = f(x)$ , is an epimorphism.

Again, as in the case of immersions, given a continuous map  $f : M \rightarrow N$  of closed manifolds, a necessary condition for the existence of a submersion  $M \rightarrow N$  homotopic to  $f$  is the existence of an epimorphism  $TM \rightarrow TN$  of tangent bundles covering the map  $f$ .

However, in contrast to the case of immersions, the existence of an epimorphism of tangent bundles does not imply the existence of a submersion. For example, since the tangent bundle of  $S^1$  is trivial, there exists a monomorphism of bundles  $TS^1 \rightarrow T\mathbb{R}^1$ , while every real valued function on  $S^1$  has a singular point.

In this sense, the h-principle for submersions of closed manifolds does not hold.

### 4. B-PRINCIPLE FOR SUBMERSIONS

The bordism principle is a bordism version of the homotopy principle. To formulate the bordism principle we will need a few more definitions.

We say that two submersions  $f_i : M_i \rightarrow N_i$ , with  $i = 1, 2$ , of closed manifolds are bordant if there is a submersion  $F : M \rightarrow N$  of a manifold  $M$  with boundary  $\partial M = M_1 \cup M_2$  into a manifold  $N$  with boundary  $\partial N = N_1 \cup N_2$  such that  $F|_{M_i} = f_i$  for  $i = 1, 2$ .

The classes of bordant submersions of manifolds of dimension  $m$  into manifolds of dimension  $n$  constitute an abelian group  $Sub(m, n)$  with addition defined in terms of representatives by taking the disjoint union of submersions.

Next we define a *stable epimorphism* of tangent bundles as a class of an epimorphism  $TM \oplus \mathbb{R}^k \rightarrow TN \oplus \mathbb{R}^k$  where  $k$  is a positive integer. If  $k_1 < k_2$ , then two epimorphisms  $\tau_i : TM \oplus \mathbb{R}^{k_i} \rightarrow TM \oplus \mathbb{R}^{k_i}$ , with  $i = 1, 2$ , represent the same stable epimorphism of tangent bundles if  $\tau_2 = \tau_1 \oplus 1_{\mathbb{R}^{k_2 - k_1}}$ , where  $1_{\mathbb{R}^{k_2 - k_1}}$  is the identity map of  $\mathbb{R}^{k_2 - k_1}$ .

Let  $[\tau_1]$  and  $[\tau_2]$  be two stable epimorphisms represented by  $\tau_i : TM_i \oplus \mathbb{R}^{k_i} \rightarrow TN_i \oplus \mathbb{R}^{k_i}$ , with  $k = 1, 2$ . Suppose that there is a manifold  $M$  with boundary  $\partial M = M_1 \sqcup M_2$ , a manifold  $N$  with boundary  $\partial N = N_1 \sqcup N_2$ , a number  $k > \max(k_1, k_2)$  and an epimorphism  $\tau : TM \oplus \mathbb{R}^k \rightarrow TN \oplus \mathbb{R}^k$  such that the epimorphisms

$$\tau|_{TM_1} : TM_1 \oplus \mathbb{R}^k \longrightarrow TN_1 \oplus \mathbb{R}^k$$

and

$$\tau|_{TM_2} : TM_2 \oplus \mathbb{R}^k \longrightarrow TN_2 \oplus \mathbb{R}^k$$

are stably equivalent to epimorphisms  $\tau_1$  and  $\tau_2$  respectively. Then we say that the stable epimorphisms  $[\tau_1]$  and  $[\tau_2]$  are *bordant*.

Let  $Epi(m, n)$  denote the bordism group of stable epimorphisms of tangent bundles covering maps of manifolds of dimension  $m$  into manifolds of dimension  $n$ . Then there is a homomorphism of groups  $Sub(m, n) \rightarrow Epi(m, n)$  that takes a class represented by a submersion  $f$  onto the class represented by an epimorphism  $df$ .

We say that the b-principle for submersions of manifolds of dimension  $m$  into manifolds of dimension  $n$  holds if the homomorphism  $Sub(m, n) \rightarrow Epi(m, n)$  is an isomorphism.

In particular, if the b-principle holds and  $f : M \rightarrow N$  is a continuous map covered by a stable epimorphism, then, the map  $f$  is bordant to a submersion.

## 5. KHAN-PRIDDY THEOREM

We say that a map of a manifold of dimension  $m$  into a manifold of dimension  $n$  has *codimension*  $n - m$ .

The bordism principle for submersions of codimension 0 turns out to be equivalent to the Khan-Priddy theorem, which asserts that there is a map  $q : K(S, 1) \rightarrow (\Omega^\infty S^\infty)_0$  inducing an isomorphism of integral homology groups, or, equivalently, of bordism groups. Here  $S$  is the group of infinite permutations, i.e., the limit  $\varinjlim \Sigma_i$  of the groups  $\Sigma_i$  of permutations on  $i$

elements; and  $(\Omega^\infty S^\infty)_0$  is the colimit of the sequence of spaces of pointed maps  $S^j \rightarrow S^j$  of degree 0.

To relate the Khan-Priddy theorem to the b-principle for submersions, let us give geometric interpretations of elements of the bordism groups of  $K(S, 1)$  and  $(\Omega^\infty S^\infty)_0$ .

An element of the oriented bordism group of  $(\Omega^\infty S^\infty)_0$  is represented by a map  $N \times S^j \rightarrow S^j$  that takes the slice  $N \times \{v\}$  over the north pole  $v \in S^j$  onto  $v$ . If necessary, we may slightly perturb this map so that it becomes transversal to the south pole  $\{*\}$  of  $S^j$ . Then the inverse image of  $\{*\}$  is a manifold  $M \subset N \times S^j$ . Furthermore, it is easy to see that the projection of  $M \subset N \times S^j$  onto the first factor  $N$  can be covered by a fiberwise isomorphism of stabilized tangent bundles. Now a straightforward argument shows that the bordism group of  $(\Omega^\infty S^\infty)_0$  is isomorphic to the bordism group of epimorphisms of tangent bundles of codimension 0.

On the other hand, the space  $K(\Sigma_i, 1)$  is a classifying space of  $i$ -sheet coverings. Hence the bordism group of  $K(\Sigma_i, 1)$  is isomorphic to the bordism group of  $i$ -sheet coverings. To obtain the bordism group of  $K(S, 1)$  we consider all coverings and identify an  $i$ -sheet covering  $f : M \rightarrow N$  with an  $(i + 1)$ -sheet covering  $f \sqcup 1_N : M \sqcup N \rightarrow N$  where  $1_N$  is the identity map of the manifold  $N$ .

Now it remains to describe the map  $q : K(S, 1) \rightarrow (\Omega^\infty S^\infty)_0$ . A point of  $K(S, 1)$  is represented by  $i$  unordered points  $p_1, \dots, p_i$  in  $\mathbb{R}^j$  for some positive integers  $i$  and  $j$ . We will assign to points  $p_1, \dots, p_i$  a map  $S^j \rightarrow S^j$  of order 0. Let  $r$  denote the minimal integer for which the disc in  $\mathbb{R}^j$  of radius  $r$  with center at  $0 \in \mathbb{R}^j$  contains all the  $i$  points. For  $i < k \leq 2i$ , pick the points  $p_k$  with coordinates  $(r + k - i, 0) \in \mathbb{R} \times \mathbb{R}^{j-1}$ . Let  $\varepsilon$  be the minimal distance between the points  $\{p_k\}$  and let  $D_k \in \mathbb{R}^j$  denote the  $j$ -disc of radius  $\frac{\varepsilon}{10}$  with center at  $p_k$ ,  $1 \leq k \leq 2i$ . Then, the desired map  $S^j \rightarrow S^j$  is the one that takes the complement to the disjoint union of the discs  $D_k$  onto the north pole  $s \in S$ , and maps each of  $D_k$  onto  $S^j$  by the composition of the factorization  $D_k \rightarrow D_k/\partial D_k = S_k$  and a homeomorphism  $S_k \rightarrow S^j$  of degree 1 if  $k \leq i$  and of degree  $-1$  otherwise.

## 6. MUMFORD CONJECTURE

The b-principle for submersions of codimension  $-2$  turns out to be equivalent to the Mumford conjecture on the cohomology ring of the moduli space of Riemann surfaces. The Mumford conjecture was recently proved by Madsen and Weiss.

Whether the b-principle for submersions of codimension  $k$  for  $k < -2$  holds or does not hold is an open problem.

## 7. WHAT IS NEXT?

The h-principle and its bordism version, the b-principle, can be formulated in a much more general setting.

For example, we may view immersions (or submersions) as smooth maps satisfying appropriate differential relations. Using the language of jet spaces, one may formulate the h-principle and b-principle for any differential relation.

It turns out that in general the bordism principle for a differential relation follows from the corresponding h-principle provided that the differential relation admits a destabilization. For example, for submersions, we require that the existence of an epimorphism of stabilized tangent bundles implies the existence of an epimorphism of genuine tangent bundles in the same bordism class. We proved that such a destabilization exists whenever the differential relation is invariant with respect to contact changes of coordinates.