Overview on the theory of currents

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1 Introduction

Geometric measure theory (GMT) is, so to speak, differential geometry generalized by the measure theory in order to deal with maps and surfaces which are not necessarily smooth. GMT is used mainly for the variational problem (for example, area-minimizing problem). In this abstract, we outline the theory of currents which is a basic notion of GMT.

2 Preliminarities

2.1 Multivectors and Covectors

Let V be a n-dimensional vector space over $\mathbb{R}$ and V^* be its dual space. For $k,\ 0 \leq k \leq n$, we define

$$I(k,n) := \{\alpha = (\alpha_1, \cdots, \alpha_k)\ ; \alpha_i \in \mathbb{Z},\ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$$

as a set of *ordered multi-indices* and for convenience we set

$$I(0,n) = \{0\}.$$

For $v_1, \cdots, v_k \in V,\ 1 \leq k \leq n$, these wedge product

$$\xi := v_1 \wedge \cdots \wedge v_k$$

is called *k-vector*. The linear space of k-vectors is denoted by $\Lambda_k V$. Similarly for $v^1, \cdots, v^k\ \in V^*,\ 1 \leq k \leq n$, these wedge product

$$\omega := v^1 \wedge \cdots \wedge v^k$$

is called *k-covector*. The linear space of k-covectors is denoted by $\Lambda^k V$. This space coincides with a dual space of $\Lambda_k V$, i.e. $\Lambda^k V = (\Lambda_k V)^*$. A k-vector

$\xi \in \Lambda_k V$ is called *simple* if it can be written as a single wedge product of vectors,

$$\xi = v_1 \wedge \cdots \wedge v_k$$

for some $v_1, \cdots, v_k \in V$. The *comass norm* of $\omega \in \Lambda^k V$ is defined by

$$||\omega|| := \sup\{< \xi, \omega > \, ; \; \xi \in \Lambda_k V, \; |\xi| \le 1, \; \xi \text{ simple}\},$$

where $< \cdot, \cdot >$ is the coupling of duality between $\Lambda_k V$ and $\Lambda^k V$ and $|\cdot|$ is the norm of k-vectors. The *mass norm* of $\xi \in \Lambda_k V$ is defined by

$$||\xi|| := \sup\{< \xi, \omega > \, ; \; \omega \in \Lambda^k V, \; ||\omega|| \le 1\}.$$

2.2 Differential forms

Let $U \subset \mathbb{R}^n$ be an open set. We denote by $\mathcal{D}^k(U)$ the space of all infinitely differentiable and compactly supported k-forms in U topologized by the usual topology which is characterized by the assertion that

$$\omega^i := \sum_{\alpha \in I(k,n)} \omega^i_\alpha dx^\alpha \longrightarrow \omega := \sum_{\alpha \in I(k,n)} \omega_\alpha dx^\alpha \qquad (i \to \infty),$$

if there is a fixed compact set $K \subset U$ such that

(i) $\operatorname{spt} \omega^i_\alpha \subset K$ for $^\forall \alpha \in I(k,n)$ and $^\forall i \in \mathbb{N}$

(ii) $\lim_{i \to \infty} D^\beta \omega^i_\alpha = D^\beta \omega_\alpha$ for $^\forall \alpha \in I(k,n)$ and every multi-index β.

We remark that k-form $\omega \in \mathcal{D}^k(U)$ is a map from U into $\Lambda^k \mathbb{R}^n$.

2.3 Rectifiable sets

A set $\mathcal{M} \subset \mathbb{R}^{n+N}$ is said to be *countably n-rectifiable* if there exist n-dimensional embedded submanifolds $\mathcal{N}_1, \mathcal{N}_2, \cdots$ and $\mathcal{N}_0 \subset \mathbb{R}^{n+N}$ with $\mathcal{H}^n(\mathcal{N}_0) = 0$ such that

$$\mathcal{M} \subset \mathcal{N}_0 \cup \bigcup_{k=1}^\infty \mathcal{N}_k.$$

Countably rectifiable sets are characterized by the property of being countable union of Lipschitz images of n-dimensional sets. More precisely, a set $\mathcal{M} \subset \mathbb{R}^{n+N}$ is countably n-rectifiable if and only if

$$\mathcal{M} \subset A_0 \cup \bigcup_{k=1}^\infty f_k(A_k)$$

where $\mathcal{H}^n(A_0) = 0$ and $f_k : A_k \to \mathbb{R}^{n+N}$ are Lipschitz maps, $A_k \subset \mathbb{R}^n$. Rectifiable sets are also characterized by the property of possessing *approximate tangent space*. Roughly, approximate tangent space is a tangent space in the sense of the measure theory.

3 Currents

Definition 3.1 (Definition of currents). *A k-dimensional current in U is a continuous linear functional on $\mathcal{D}^k(U)$. The space of k-dimensional currents in U is denoted by $\mathcal{D}_k(U)$.*

Definition 3.2 (Weak convergence of currents). *Let $T_j, T \in \mathcal{D}_k(U)$. We say that the sequence $\{T_j\}_{j=1}^{\infty}$ converges weakly to T,*

$$T_j \rightharpoonup T \qquad (j \to \infty),$$

if

$$T_j(\omega) \longrightarrow T(\omega) \qquad \text{for } ^\forall\omega \in \mathcal{D}^k(U).$$

Definition 3.3 (Boundary of currents). *The* boundary *of $T \in \mathcal{D}_k(U)$ is defined as the $(k-1)$-current*

$$\partial T(\eta) := T(d\eta) \qquad \text{for } ^\forall\eta \in \mathcal{D}^{k-1}(U),$$

where $d\eta$ is an exterior differential of η. And we set

$$\partial T = 0 \qquad \textit{if } T \in \mathcal{D}_0(U).$$

Definition 3.4 (Mass norm). *Let $U \subset \mathbb{R}^n$ and $V \subset U$ be an open sets, and let $T \in \mathcal{D}_k(U)$. The* mass norm *of T in V is defined by*

$$\mathbf{M}_V(T) := \sup\{T(\omega)\ ;\ \omega \in \mathcal{D}^k(U),\ \operatorname{spt}\omega \subset V,\ ||\omega|| \le 1 \text{ for } ^\forall x \in U\}.$$

If $V = U$ we shall simply write $\mathbf{M}(T)$ instead of $\mathbf{M}_V(T)$, and we set

$$\mathcal{M}_k(U) := \{T \in \mathcal{D}_k(U)\ ;\ \mathbf{M}(T) < \infty\}$$
$$\mathcal{M}_{k,\mathrm{loc}}(U) := \{T \in \mathcal{D}_k(U)\ ;\ \mathbf{M}_V(T) < \infty \text{ for } ^\forall V \subset\subset U\}.$$

From the definition of mass norm we readily infer

Proposition 3.5 (Lower semicontinuity of the mass norm). *Let $T_j, T \in \mathcal{D}_k(U)$. If $T_j \rightharpoonup T$ then for any $V \subset U$, V open,*

$$\mathbf{M}_V(T) \le \liminf_{j\to\infty} \mathbf{M}_V(T_j).$$

And from the compactness theorem for Radon measures

Proposition 3.6 (Compactness-closure theorem). *Let* $\{T_j\}_{j=1}^{\infty} \subset \mathcal{M}_{k,\mathrm{loc}}(U)$ *be a sequence satisfying*

$$\sup_j \mathbf{M}_V(T_j) < \infty \qquad \text{for } {}^{\forall}V \subset\subset U.$$

Then there exists a subsequence $\{T_{j'}\}_{j'=1}^{\infty} \subset \{T_j\}$ *and* $T \in \mathcal{M}_{k,\mathrm{loc}}(U)$ *such that*

$$T_{j'} \rightharpoonup T \qquad (j' \to \infty).$$

Moreover

$$\mathbf{M}(T) \leq \liminf_{j' \to \infty} \mathbf{M}(T_{j'}) < \infty$$

if the mass norms of $T_{j'}$ *are equibounded.*

We define the class of *integer multiplicity rectifiable currents.* This class is very important in the theory of currents.

Definition 3.7 (Rectifiable currents). *Let* U *be a open set in* $\mathbb{R}^n$*. We say that a current* $T \in \mathcal{D}_k(U)$ *is* rectifiable *if and only if there exists an* $\mathcal{H}^k$*-measurable and countably* k*-rectifiable set* $\mathcal{M} \subset U$*, an* $\mathcal{H}^k$*-measurable and locally* $\mathcal{H}^k\lfloor_{\mathcal{M}}$*-summable function* $\theta : \mathcal{M} \to \mathbb{R}$*, and an* $\mathcal{H}^k$*-measurable map* $\xi : \mathcal{M} \to \Lambda_k \mathbb{R}^n$ *with* $||\xi(x)|| = 1$ $\mathcal{H}^k\lfloor_{\mathcal{M}}$-a.e.$x$ *such that*

$$T(\omega) = \int_{\mathcal{M}} < \xi(x), \omega(x) > \theta(x)\, d\mathcal{H}^k(x)$$

and moreover $\xi(x)$ *is a* k*-vector associated to the approximate tangent space of* $\mathcal{M}$ *for* $\mathcal{H}^k$-a.e.x*. If moreover* θ *is integer-valued, then we say that* T *is* integer multiplicity rectifiable*. The class of integer multiplicity rectifiable* k*-currents in* U *is denoted by* $\mathcal{R}_k(U)$*.*

4 Important theorems for currents

For integer multiplicity rectifiable currents it holds that

Theorem 4.1 (Closure theorem). *Let* $\{T_j\}_{j=1}^{\infty} \subset \mathcal{R}_k(U)$ *be a sequence of integer multiplicity rectifiable* k*-currents in some open set* $U \subset \mathbb{R}^n$ *satisfying*

$$\sup_j \{\mathbf{M}_V(T_j) + \mathbf{M}_V(\partial T_j)\} < \infty \qquad \text{for } {}^{\forall}V \subset\subset U : \textit{open}$$

and weakly converging to some current $T \in \mathcal{D}_k(U)$*,* $T_j \rightharpoonup T$*. Then* $T \in \mathcal{R}_k(U)$*.*

This theorem together with the compactness theorem for general currents (Proposition 3.6) yields

Theorem 4.2 (Compactness Theorem). *Let* $\{T_j\}_{j=1}^{\infty} \subset \mathcal{R}_k(U)$ *be a sequence satisfying*

$$\sup_j \{\mathbf{M}_V(T_j) + \mathbf{M}_V(\partial T_j)\} < \infty \qquad \text{for } {}^{\forall}V \subset\subset U : \textit{open}.$$

Then there exists a subsequence $\{T_{j'}\}_{j'=1}^{\infty}$ *of* $\{T_j\}$ *and* $T \in \mathcal{R}_k(U)$ *such that*

$$T_{j'} \rightharpoonup T \qquad (j' \to \infty)$$

References

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