

A CHARACTERIZATION OF PSEUDO-EINSTEIN REAL HYPERSURFACES

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Let M be a Kähler manifold with almost complex structure J and metric g . Holomorphic sectional curvature is defined by $K(p) = g(R(X, JX)JX, X)$ where a unit vector X in $p \in T_x(M)$. If $K(p)$ is a constant for all J -invariant planes $p \in T_x(M)$ and for all points $x \in M$, then M is called a complex space form. We study real hypersurfaces of complex space forms with nonzero constant holomorphic sectional curvature under some condition on the Ricci tensor.

In 1968, Nomizu studied hypersurfaces in Euclidean spaces so that the curvature tensor R satisfies $R(X, Y)R = 0$ (locally semi-symmetric), which is a condition weaker than $\nabla R = 0$ (locally symmetric). The relationship between this condition and the condition $R(X, Y)S = 0$ (Ricci semi symmetric) was studied by Ryan [7] for hypersurfaces in real space forms, where S denotes the Ricci tensor. For real hypersurfaces of complex space forms, Kimura and Maeda [3] (see also [2]) proved that there are no Ricci semi symmetric real hypersurfaces in CP^n , $n \geq 3$.

In this talk, we study a real hypersurface M satisfying the condition that

$$\begin{aligned} g((R(X, Y)S)Z, W) &= 0 \text{ for any tangent vector fields} \\ X, Y, Z \text{ and } W \text{ orthogonal to } \xi &:= JN, \end{aligned} \quad (*)$$

where N is the unit normal of M in a complex space form $M^n(c)$ with complex structure J .

Let g be the induced metric of M . We define a tensor field ϕ of type (1,1) and a 1-form η by

$$JX = \phi X + \eta(X)N,$$

where ϕX is the tangential part of JX . (ϕ, ξ, η, g) defines an almost contact metric structure on M .

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the one induced on M . Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

where A is the shape operator of M .

The equation of Gauss is given by

$$\begin{aligned} R(X, Y)Z &= c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &\quad - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &\quad + g(A Y, Z)A X - g(A X, Z)A Y, \end{aligned}$$

which implies that the Ricci tensor S of M is given by

$$SX = (2n + 1)cX - 3c\eta(X)\xi + hAX - A^2X, \quad (1)$$

where $h(:= \text{tr}A)$ is the mean curvature of M . Moreover, the scalar curvature $r(:= \text{tr}S)$ of M is given by

$$r = 4c(n^2 - 1) + h^2 - \text{tr}A^2.$$

If the Ricci tensor S of M satisfies $g(SX, Y) = ag(X, Y)$ where a is a constant, then M is called an Einstein real hypersurface. If S satisfies $g(SX, Y) = ag(X, Y) + b\eta(X)\eta(Y)$ for some constants a and b , then M is called a pseudo-Einstein real hypersurface. Kon [4] classified pseudo-Einstein real hypersurfaces of a complex projective space and proved that any real hypersurface of a complex projective space is not Einstein, which is improved by Cecil and Ryan [1] in case that a and b are functions. Montiel [5] proved the corresponding results in the case that the ambient manifold is a complex hyperbolic space.

Lemma. *Let M be a real hypersurface satisfying the condition (*) of a complex space form $M^n(c)$, $c \neq 0$, $n \geq 3$. Then we have*

$$g(SX, Y) = \frac{1}{2n - 2}(r - g(S\xi, \xi))g(X, Y),$$

for any vector fields X and Y orthogonal to ξ .

Using this, we prove the following

Theorem. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$, $n \geq 3$. M satisfies $g((R(X, Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ if and only if M is pseudo-Einstein.*

Proof. We suppose that M satisfies the condition (*). We can choose a local field of orthonormal frames $\{X_1, \dots, X_{2n-2}, \xi\}$ of M such that the shape operator A is represented by a matrix form

$$A = \begin{bmatrix} \lambda_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_{2n-2} & h_{2n-2} \\ h_1 & \cdots & h_{2n-2} & \alpha \end{bmatrix}.$$

Using Lemma, we see that at most one h_i does not vanish. Thus we can assume that $h_i = 0$ for $i \geq 2$. We set $a = g(SX_i, X_i)$. Then we have

$$\begin{aligned} SX_1 &= aX_1 + h_1(h - \lambda_1 - \alpha)\xi, \\ SX_i &= aX_i \quad (i = 2, \dots, 2n-2), \\ S\xi &= h_1(h - \lambda_1 - \alpha)X_1 + ((2n-2)c + \alpha h - h_1^2 - \alpha^2)\xi. \end{aligned}$$

From these equations and the equation of Gauss, for any $j \geq 2$, we obtain

$$\begin{aligned} 0 &= g(R(X_1, X_j)SX_1, X_j) - g(SR(X_1, X_j)X_1, X_j) \\ &= h_1(h - \lambda_1 - \alpha)g(R(X_1, X_j)\xi, X_j). \end{aligned}$$

By the equation of Gauss, we have

$$g(R(X_1, X_j)\xi, X_j) = -h_1\lambda_j.$$

These equations imply that $h_1(h - \lambda_1 - \alpha) = 0$. So we see that M is pseudo-Einstein.

Conversely, if M is pseudo-Einstein, we have $SZ = aZ + b\eta(Z)\xi = aZ$ and $SW = aW$ for any tangent vectors Z and W orthogonal to ξ . Then we have

$$g((R(X, Y)S)Z, W) = g(R(X, Y)SZ, W) - g(SR(X, Y)Z, W) = 0.$$

q.e.d.

References

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