Hom stacks and Picard stacks

Masao AOKI

1 Introduction

The concept of *algebraic stack* is a generalizations of the concept of scheme, in the same sense that the concept of scheme is a generalization of the concept of variety. The study of algebraic stacks is motivated by problems of moduli. In many moduli problems, the moduli functor we want to study is not representable by a scheme. But if we enlarge the category of schemes, we can construct "moduli stacks", which carries all the information we want.

Moduli stacks are now an important tool in the Gromov-Witten theory. Some people use algebraic stacks to study a variety with quotient singularity, which is approximated by a smooth algebraic stack. Algebraic stacks are also used in the study of non-abelian cohomology and non-abelian Hodge theory.

In the first half of this talk I talk what is algebraic stacks, and why we need them. In the second half I introduce the Hom stacks and the Picard stacks.

2 Algebraic Stacks

2.1 Functor of points – Grothendieck's viewpoint

The origin of algebraic geometry is the study of varieties. A variety is, roughly speaking, the set of zero points of polynomials. However, the set of zero points depends on the field (or ring) in which we consider "zero points". For instance, the set zero points of the polynomial $X^n + Y^n - 1$ is:

- a curve on a plane, if we consider zero points in the field $\mathbb{R}$.
- a Riemann surface of genus $n(n+1)/2$, in $\mathbb{C}$
- a finite set of points, in $\mathbb{Q}$
- a curve and its tangent lines on each point, in the ring $\mathbb{R}[\epsilon]/(\epsilon^2)$

The idea of Grothendieck is to consider the functor which corresponds a ring R with the set of zero points in R, or the set of R-valued points. In the words of schemes, the set of R-valued points of a scheme (or a variety) X is identified with the set of morphisms from $\operatorname{Spec} R$ to X.

Now we have a natural question: when a functor X : (Rings) $\to$ (Sets) is representable by a variety (or a scheme)? No definite condition is known, but there is a consequent:

Theorem (étale descent). *Let X be a scheme and $\{U_i \xrightarrow{p_i} U\}_{i \in I}$ an étale covering of an affine scheme. Then*

- *The map $X(U) \to \prod_i X(U_i)$ is injective.*
- *For any $\{x_i\} \in \prod_i X(U_i)$, there exists $y \in X(U)$ such that $y|_{U_i} = x_i$ $(\forall i \in I)$ if and only if $x_i|_{U_i \times_U U_j} = x_j|_{U_i \times_U U_j}$ $(\forall i, j \in I)$.*

In other words, if we endow the category of affine schemes with étale topology, the functor X is a sheaf.

2.2 Problems of moduli

A moduli space is a space which parametrizes a certain kind of (equivalence classes of) objects, such as curves, surfaces, line bundles and so on. With the viewpoint of Grothendieck, a problem of moduli is formulated as follows. Let F be a functor from (Rings) to (Sets) defined by

$$F(U) = \{\text{objects defined over } U\}.$$

When is the functor F represented by a scheme?

There are some examples of moduli spaces. Let X be a scheme. Under certain conditions, the functors

$$\begin{aligned} \mathrm{Hilb}_X(U) &= \{\text{subschemes of } X \times U, \text{ proper and flat over } U\} \\ \mathrm{Pic}_X(U) &= \{\text{line bundles on } X \times U\}/\text{equivalence relation} \end{aligned}$$

are representable by schemes, namely the Hilbert schemes and the Picard schemes.

Unfortunately, some moduli functors – for example moduli of curves and moduli of vector bundles on a scheme – are not representable by schemes or do not have good properties we want. This is because curves and vector bundles have nontrivial automorphisms and the functor to the category of sets loses informations of automorphisms.

We replace the set of points by the "category (groupoid) of points". A groupoid is a category whose morphisms are all isomorphisms. To give a groupoid of points is equivalent to give a set of points and automorphism groups of each points. A moduli functor is replaced by 2-functors, for example

$$\begin{aligned} \mathscr{M}_g(U) &= \text{the category of stable curves of genus } g \text{ over } U \\ \mathscr{M}_X(U) &= \text{the category of vector bundles on } X \times U \end{aligned}$$

These 2-functors have some good properties so that we can treat them like schemes.

2.3 What are algebraic stacks?

We define two generalizations of schemes – algebraic spaces and algebraic stacks.

Recall the definition of schemes. A scheme is a ringed space (a topological space and a sheaf of rings on it) (X, O_X) which is locally isomorphic to an affine scheme $\operatorname{Spec} R$.

An **algebraic space** is a functor $X : (\text{Rings}) \to (\text{Sets})$ which satisfies the following conditions (plus some minor conditions):

- X is a sheaf on the category of affine schemes with étale topology.
- There is an étale surjection $P : X^0 \to X$ from a scheme X^0.

The first condition means that X looks like a ringed space. We can consider sheaves on X and the structural sheaf O_X. The second condition means that X is locally (in étale topology) isomorphic to a scheme.

A 2-functor $\mathscr{X} : (\text{Rings}) \to (\text{groupoids})$ relates to any ring R the category of R-valued points $\mathscr{X}(R)$ "functorially" in 2-categorical sense. An **algebraic stack (champ algébrique**[1]**)** is a 2-functor $\mathscr{X} : (\text{Rings}) \to (\text{groupoids})$ which satisfies the following conditions (plus some minor conditions):

- $\mathscr{X}$ satisfies a 2-categorical analogue of the descent condition.
- There is a smooth surjection $P : X^0 \to \mathscr{X}$ from a scheme X^0.

The morphism P is called a presentation of $\mathscr{X}$. If P is étale, $\mathscr{X}$ is called a Deligne-Mumford stack.

The moduli stack of curves $\mathscr{M}_g$ is a Deligne-Mumford stack and the moduli stack of verctor bundles $\mathscr{M}_X$ is an algebraic stack. We can generalize many properties and concepts related to schemes to algebraic stacks.

3 Hom stacks and Picard Stacks

3.1 Picard stacks

We generalize the concept of Picard schemes to algebraic stacks.

Let S be a noetherian noetherian scheme and $\mathscr{X}$ an algebraic stacks over S. The **Picard stack**[2] $\mathscr{P}ic_{\mathscr{X}}$ is a stack over S defined by

$$\mathscr{P}ic_{\mathscr{X}}(U) = \text{ the category of line bundles on } \mathscr{X} \times_S U$$

A line bundle on $\mathscr{X}$ is identified with a $\mathbb{G}_m$-bundle on $\mathscr{X}$. On the other hand, there is an algebraic stack $B\mathbb{G}_m = [S/\mathbb{G}_m]$, whose category of points over U is the category of $\mathbb{G}_m$-bundles over U. So we can identify a line bundle on $\mathscr{X}$ with a morphism from $\mathscr{X}$ to the algebraic stack $B\mathbb{G}_m$.

Thus the representability of Picard stacks follows from that of another kind of stacks — Hom stacks.

[1] French

[2] This word may be confusing. Deligne used the word "champ de Picard" for a stack with commutative monoid structure. As we see later, the Picard stack in our sense is an example of "champ de Picard" in Deligne's sense.

3.2 Hom stacks

Let S be a noetherian noetherian scheme and $\mathscr{X}$ and $\mathscr{Y}$ algebraic stacks over S. Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are of finite presentation.

We define the **Hom stack** $\mathscr{HOM}(\mathscr{X}, \mathscr{Y})$ as follows:

$$\mathscr{HOM}(\mathscr{X}, \mathscr{Y})(U) = \mathrm{HOM}(\mathscr{X} \times_S U, \mathscr{Y} \times_S U)$$

The right hand side is the category whose objects are 1-morphisms between algebraic stacks, and whose morphisms are 2-morphisms.

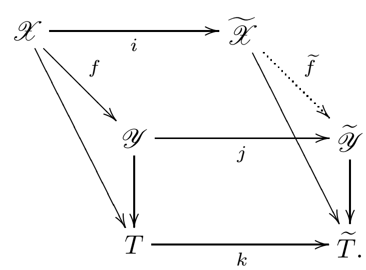
Theorem 1. *If $\mathscr{X}$ is proper and flat over S, then $\mathscr{HOM}(\mathscr{X}, \mathscr{Y})$ is an algebraic stack in Artin's sense.*

Corollary 2. *If $\mathscr{X}$ is proper and flat over S, then $\mathscr{P}ic_{\mathscr{X}} = \mathscr{HOM}(\mathscr{X}, B\mathbb{G}_m)$ is an algebraic stack in Artin's sense.*

If $\mathscr{X}$ and $\mathscr{Y}$ are schemes, We can identify a morphism $f : \mathscr{X} \to \mathscr{Y}$ with the graph of f, which is a closed subscheme of $\mathscr{X} \times_S \mathscr{Y}$. Thus the Hom scheme $\mathscr{HOM}(\mathscr{X}, \mathscr{Y})$ is identified with a subscheme of the Hilbert scheme $\mathrm{Hilb}_{\mathscr{X} \times_S \mathscr{Y}}$. However, in the case of algebraic stacks, we have no "Hilbert stacks" yet, and the graph of a morphism f is not a closed substack of $\mathscr{X} \times_S \mathscr{Y}$.

To prove Theorem 1 we use the Artin's criterion [Ar]. The key of the proof is the deformation theory of morphisms of algebraic stacks.

Let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic stacks over a noetherian scheme T, $f : \mathscr{X} \to \mathscr{Y}$ a morphism over T. Consider the diagram of solid arrows.



Here i, j and k are closed immersions defined by a square-zero ideals. Then we have the following theorem. This is a generalization of the deformation theory of schemes.

Theorem 3.

1. *There is an obstruction $o \in \mathrm{Ext}^1(Lf^* L_{\mathscr{Y}/T}, I)$, and a deformation $\widetilde{f}$ exists if and only if $o = 0$.*
2. *If $o = 0$, then the set of isomorphism classes of f is a torsor under the group $\mathrm{Ext}^0(Lf^* L_{\mathscr{Y}/T}, I)$.*
3. *The 2-automorphism group of a deformation $\widetilde{f}$ is isomorphic to $\mathrm{Ext}^{-1}(Lf^* L_{\mathscr{Y}/T}, I)$.*

Here $L_{\mathscr{Y}/T}$ is the cotangent complex induced by Laumon and Moret-Bailly [LM].

3.3 Group stacks

Group stacks are 2-categorical analogue of group schemes. A group structures on an algebraic stack $\mathcal{G}$ over S is given by morphisms

$$\begin{aligned} m &: \mathcal{G} \times_S \mathcal{G} \to \mathscr{X} \\ e &: S \to \mathcal{G} \\ i &: \mathcal{G} \to \mathcal{G} \end{aligned}$$

and 2-isomorphisms

$$m(\mathrm{id} \times m) \Rightarrow m(m \times \mathrm{id}), \qquad \text{(associativity)}$$
$$m(\mathrm{id} \times e) \Rightarrow \mathrm{id}, \qquad m(e \times \mathrm{id}) \Rightarrow \mathrm{id}, \qquad \text{(unit)}$$
$$m(\mathrm{id} \times i) \Rightarrow e, \qquad m(i \times \mathrm{id}) \Rightarrow e \qquad \text{(inverse)}$$

with certain commutativities.

The Picard stack $\mathscr{P}ic_{\mathscr{X}}$ has a group structure given by tensor products. Another example of group stacks is the stack of automorphisms $\mathscr{A}ut(\mathscr{X})$, which is a substack of the Hom stack $\mathscr{HOM}(\mathscr{X}, \mathscr{X})$.

The author intends to study the theory of group stacks, which generalizes the theory of group schemes and will be applied on many kinds of moduli problems.

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青木 昌雄
Department of Mathematics, Kyoto University
`aoki@math.kyoto-u.ac.jp`
`http://www.math.kyoto-u.ac.jp/~aoki/`