

ON THE COLORED JONES POLYNOMIAL AND THE VOLUME CONJECTURE

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The colored Jones polynomial is one of link invariants which includes the Jones polynomial. It is obtained from the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ and its irreducible representation.

Let L be an m -component oriented link and N_i ($i = 1, \dots, m$) be a positive integer. We can calculate the colored Jones polynomial of L denoted by $J_{\{N_1, \dots, N_m\}}(L; q)$ as follows. First we associate the i -th component of L with the positive integer N_i and present L into a (1,1)-tangle T by cutting a component. Next we label each edge of T in the component associated with N_i an element in $\{0, 1, \dots, N_i - 1\}$. Here two edges containing end points of T are labeled 0. (Figure 1)

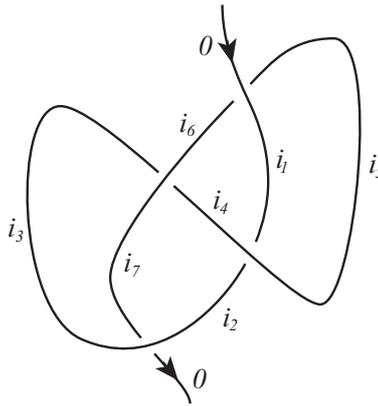
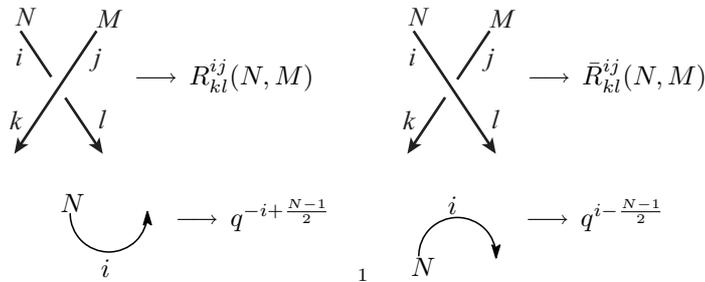


FIGURE 1. (1,1)-tangle decomposition of the figure-eight knot 4_1 .

Next We assign each crossing and each edge containing a local maximal point or minimal point to some values as follows.



Here $R_{kl}^{ij}(N, M)$ and $\bar{R}_{kl}^{ij}(N, M)$ are given by

$$R_{kl}^{ij}(N, M) = \sum_{n=0}^{\min\{j, M-1-j\}} \delta_{l, i+n} \delta_{k, j-n} (-1)^n \frac{(q)_{i+n} (q)_{M-1+n-j}}{(q)_n (q)_i (q)_{M-1-j}} \\ \times q^{-\frac{n^2}{2} - n(\frac{N+M}{4} + i - j) + (i - \frac{N-1}{2})(j - \frac{M-1}{2})},$$

$$\bar{R}_{kl}^{ij}(N, M) = \sum_{n=0}^{\min\{i, N-1-i\}} \delta_{l, i-n} \delta_{k, j+n} \frac{(q)_{j+n} (q)_{M-1+n-i}}{(q)_n (q)_j (q)_{M-1-i}} \\ \times q^{\frac{-3N+M+2}{4}n - (i - \frac{N-1}{2})(j - \frac{M-1}{2})},$$

where $(q)_n = \prod_{k=1}^n (1-q)^k$. After multiplying all elements obtained from T as above we sum over all indices. Then we have $J_{\{N_1, \dots, N_m\}}(L; q)$.

For example the colored Jones polynomial of the figure-eight knot is

$$J_N(4_1; q) = \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7=0}^{N-1} \bar{R}_{i_6 i_1}^{0 i_5}(N, N) R_{i_7 i_4}^{i_3 i_6}(N, N) \bar{R}_{i_2 i_5}^{i_4 i_1}(N, N) R_{i_3 0}^{i_7 i_2}(N, N) q^{i_3 - i_5} \\ = \sum_{i=0}^{N-1} q^{iN} \prod_{k=1}^i (1 - q^{-N+k}) (1 - q^{-N-k}).$$

If all components are colored by 2 then the colored Jones polynomial coincides with the Jones polynomial. (see [9])

We have no explicit geometric interpretation of the quantum invariant but the volume conjecture implies some relation between the colored Jones polynomial and the geometric invariant of knot exterior. R. Kashaev defined certain link invariants by constructing the R-matrix and he conjectured in [3] that for hyperbolic knots the asymptotic behavior of his link invariant determines their hyperbolic volumes. H. Murakami and J. Murakami proved in [7] that Kashaev's invariant coincides with the N colored Jones polynomial evaluated at $\exp(2\pi\sqrt{-1}/N)$ which is also coincides with the Akutsu-Deguchi-Ohtsuki invariant, and they generalized Kashaev's conjecture to the volume conjecture.

Let $J_N(L; q)$ be the colored Jones polynomial of a link L whose all components are colored by N and it is normalized such that the value of the trivial knot is 1. Then the volume conjecture is described as follows.

Conjecture 1. *Let K be a knot in the three-sphere S^3 . Then*

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = v_3 \|S^3 - K\|,$$

where $\|S^3 - K\|$ is the simplicial volume of $S^3 - K$ and v_3 is the volume of the ideal regular tetrahedron. If K is a hyperbolic knot which means its complement admits the hyperbolic structure, then $v_3 \|S^3 - K\|$ is equal to the hyperbolic volume of $S^3 - K$.

This conjecture holds for the figure-eight knot, torus knots [4] and the Borromean rings [1]. Also the relation between Kashaev's R-matrix and the geometric structure of the knot exterior was considered in [10]. Recently we considered some generalization of the volume conjecture. We expect that the asymptotic behavior of the colored Jones polynomial evaluated at $\exp(2\pi\alpha\sqrt{-1}/N)$ determines the

some geometric invariant of 3-manifolds obtained from the knot. In fact for the figure-eight knot it was proved in [6] that the asymptotic behavior of the colored Jones polynomial evaluated at $\exp(2\pi r\sqrt{-1}/N)$ for a real number r determines the volume of the cone-manifold with singularity along the figure-eight knot. Moreover it was proved in [8] that the asymptotic behavior of the colored Jones polynomial of the figure-eight knot evaluated at $\exp(2\pi\alpha\sqrt{-1}/N)$ for complex number α determines the Neumann-Zagier function.

We consider the colored Jones polynomial of the Borromean rings B evaluated at $\exp(2\pi r\sqrt{-1}/N)$ for a fixed real number r . Let $\Lambda(z)$ be the Lobachevsky function defined as $\Lambda(z) = -\int_0^z \log|2\sin t|dt$. We define two functions $V_1(r)$ and $V_2(r)$ as follows.

$$\begin{aligned} V_1(r) &= 2(3(\Lambda(\pi r + \theta) - \Lambda(\pi r - \theta)) - 4\Lambda(\theta + \pi/2) - 2\Lambda(\theta)), \\ V_2(r) &= 2(-3(\Lambda(\psi + \pi r) + \Lambda(\psi - \pi r)) + 4\Lambda(\psi + \pi/2) + 2\Lambda(\psi)), \end{aligned}$$

where $\theta = \theta(r)$ and $\psi = \psi(r)$, $0 < \theta, \psi < \frac{\pi}{2}$ are principal parameters defined by conditions

$$\begin{aligned} T &= \tan \theta, T^4 - (3 \tan^2(\pi r) + 1)T^2 - \tan^6(\pi r) = 0, \\ T' &= \tan^2 \psi, \\ T'^3 - 3 \tan^2(\pi r)T'^2 + (\tan^6(\pi r) + 6 \tan^4(\pi r) + 3 \tan^2(\pi r) + 1)T' - \tan^6(\pi r) &= 0. \end{aligned}$$

Let r_1 and r_2 be solutions of the equation $V_1(r) = V_2(r)$ and $0 < r_1 < r_2 < 1$. Then we have the following theorem.

Theorem 2. *Let r be the irrational number satisfying $r_2 < r < 1 + r_1$. Then*

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(B; \exp(2\pi r\sqrt{-1}/N))|}{N} = \frac{1}{r} V_1(r).$$

Here $V_1(r)$ coincides with the hyperbolic volume of the cone manifold whose underlying space is the three-sphere and whose singular set consists of three components of the Borromean rings with cone angles $2\pi|1-r|$, $2\pi|1-r|$ and $2\pi|1-r|$. [5]

Next we consider some extension of this theorem. We denote $J_{\{N_1, N_2, N_3\}}(B; q)$ by the colored Jones polynomial of B whose components are colored by N_1 , N_2 and N_3 . We consider $J_{\{N, aN, bN\}}(B; q)$ for positive integer a and b . By some numerical computation we can see that the asymptotic behavior of $J_{\{N, aN, bN\}}(B; q)$ also determine the volume of the cone manifold with cone angles $2\pi|1-r|$, $2\pi|1-ar|$ and $2\pi|1-br|$.

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