Vertex operator algebra and McKay's E_8 observation on the Monster

Hiroshi Yamauchi

Graduate School of Mathematical Sciences, University of Tokyo

3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 JAPAN

e-mail: yamauchi@ms.u-tokyo.ac.jp

February 15, 2006

Abstract

As one of mysterious properties of the Monster simple group, J. McKay observed that there exists an interesting connection between the 2A-conjugacy class of the Monster and the (extended) E_8 Dynkin diagram. Monster, the largest sporadic finite simple group, can be defined by the symmetry group of the moonshine vertex operator algebra. In this talk we will report on results of our research to find the structure of E_8 diagram inside the moonshine vertex operator algebra. This is a joint work with Ching Hung Lam and Hiromichi Yamada.
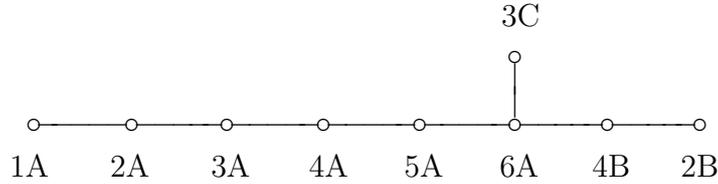
1   McKay's E_8-observation on the Monster

After the classification of finite simple groups, we found the Monster as the largest sporadic finite simple group $\mathbb{M}$. The Monster $\mathbb{M}$ has many mysterious properties; probably the most famous example is known to as the moonshine phenomenon or the monstrous moonshine [CN, B]. Another example is McKay's E_8-observation on the Monster, which is the main topic of this talk. Let us explain McKay's observation briefly. The Monster has two conjugacy classes of involutions, called 2A and 2B-conjugacy classes [ATLAS]. Let $g, h \in \mathbb{M}$ be two involutions of 2A-conjugacy class. Then the order of a product gh is known to be less than or equal to 6, which is sometimes referred to as the 6-transposition property, and the conjugacy class of the product falls into one of the following [ATLAS, C]:

$$\text{1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, or 3C,}$$

where the name pX is used in the way that p denotes the order and the letter X is labeled to distinguish classes having the same order (cf. [ATLAS]). In [Mc], J. McKay pointed

out that the numbers above can be attached to the extended affine E_8 diagram as its labels in the following way:

3C

1A 2A 3A 4A 5A 6A 4B 2B

This is McKay's E_8-observation on the Monster. By this observation, it seems to exist some relations between the 2A-conjugacy class of the Monster and E_8 Dynkin diagram. Since the Monster is known to be the automorphism group of the moonshine vertex operator algebra $V^\natural$ constructed by Frenkel et al. in [FLM], it is expected that the trick of this mystery may be revealed by finding suitable E_8 structure inside $V^\natural$. Our approach is in fact based on this point of view. In [LYY1, LYY2] we introduced an idea to explain the mystery above by using Virasoro vertex operator algebras and the lattice vertex operator algebra associated to $\sqrt{2}E_8$-lattice, which we believe to be the key of elucidation of the observation.

   To explain our works, one has to have some basic knowledge of a vertex operator algebra. However, I do not have enough pages to explain all in this article. So I decided to explain the main idea of our results in the forthcoming talk and I would like to devote the rest of this article to introduction to vertex operator algebras for non-experts since the most of participants would be unfamiliar with vertex operator algebra theory. After reading this introduction, I expect the reader would obtain enough ability to read the introduction of [LYY1], from which one may grasp our main idea.

2   Introduction to vertex operator algebras

We give a brief introduction of definition of a vertex operator algebra.

2.1   Notation

First, we fix our terminology and notation. Let V be a linear space, z be an indeterminant. Then we define the following linear spaces:

$$
\begin{aligned}
V[\![z, z^{-1}]\!] &:= \{\textstyle\sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V\}, \\
V((z)) &:= \{\textstyle\sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V,\ v_n = 0 \ \text{if}\ n \ll 0\}.
\end{aligned}
$$

For $f(z) = \sum_{n \in \mathbb{Z}} v_n z^n \in V[\![z, z^{-1}]\!]$, we define the *formal residue* by $\mathrm{Res}_z f(z) := v_{-1}$, the coefficient of z^{-1} in $f(z)$. For $n \in \mathbb{Z}$, we define the formal binomial expansion by

$$(z + w)^n := \sum_{i \geq 0} \binom{n}{i} z^{n-i} w^i.$$

Namely, $(z + w)^n$ is expanded in positive powers of the second variable w. Note that $(z + w)^n \neq (w + z)^n$ unless $n \geq 0$.

2.2 Fields

Let M be a linear space over $\mathbb{C}$. A formal power series $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}(M)[\![z, z^{-1}]\!]$ is called a *field on M* if it satisfies $a(z)v \in M((z))$ for any $v \in M$. The space of fields on M forms a $\mathbb{C}$-linear space which we denote by $\mathcal{F}(M)$. It is clear that $\mathbb{1}_M(z) := \mathrm{id}_M$ is a field on M. We call $\mathbb{1}_M(z)$ the *vacuum field* on M.

2.3 Normal product

For $a(z), b(z) \in \mathcal{F}(M)$, its composition $a(z)b(z)$ does not be a field on M in general. However, we can define the following product on $\mathcal{F}(M)$. For $n \in \mathbb{Z}$, we define the *n-th normal product $a(z) \circ_n b(z)$* of $a(z)$ and $b(z)$ by means of

$$a(z) \circ_n b(z) := \mathrm{Res}_{z_1}\{(z_1 - z)^n a(z_1)b(z) - (-z + z_1)^n b(z)a(z_1)\}. \tag{2.1}$$

One can check that $a(z) \circ_n b(z)$ is a field on M. Therefore, $\mathcal{F}(M)$ has infinitely many products $\circ_n : \mathcal{F}(M) \otimes \mathcal{F}(M) \to \mathcal{F}(M)$ for $n \in \mathbb{Z}$.

Remark 2.1. One can directly see that (i) $a(z) \circ_{-1} \mathbb{1}_M(z) = \mathbb{1}_M(z) \circ_{-1} a(z) = a(z)$, (ii) $a(z) \circ_{-2} \mathbb{1}_M(z) = \partial_z a(z)$, where $\partial_z f(z)$ denotes the formal differential of $f(z)$.

2.4 Locality

Let $a(z), b(z) \in \mathcal{F}(M)$. We say $a(z)$ and $b(z)$ are *local* if there is $N \in \mathbb{Z}$ such that

$$(z_1 - z_2)^N a(z_1)b(z_2) = (-z_2 + z_1)^N b(z_2)a(z_1) \tag{2.2}$$

holds in $\mathrm{End}(M)[\![z_1^{\pm 1}, z_2^{\pm 1}]\!]$, and we write $a(z) \sim b(z)$. Among the integers N satisfying (2.2), the minimum one is called the *order of locality between $a(z)$ and $b(z)$* and we denote it by $N(a, b)$. Clearly $N(a, b) = N(b, a)$ and we note that $a(z) \circ_{N(a,b)+i} b(z) = 0$ for any $i \geq 0$.

Remark 2.2. It is not true that $a(z) \sim a(z)$ for any $a(z) \in \mathcal{F}(M)$. It is also not true that $a(z) \sim b(z)$ and $b(z) \sim c(z)$ implies $a(z) \sim c(z)$.

2.5 Dong's lemma

Lemma 2.3. *([L, K]) If $a(z), b(z), c(z) \in \mathcal{F}(M)$ are pair-wise local, then so are $a(z)$, $b(z)$, $c(z)$ and $a(z) \circ_n b(z)$ for any $n \in \mathbb{Z}$. In particular, $a(z) \circ_n b(z) \sim a \circ_n b(z)$ if both $a(z) \sim a(z)$ and $b(z) \sim b(z)$.*

The lemma above is known as Dong's lemma. By this lemma, mutually local fields forms a subspace under the normal products. This fact characterizes the notion of a vertex algebra.

2.6 Vertex algebras

Definition 2.4. A *vertex algebra* is a subspace $\mathfrak{A}$ of the space $\mathcal{F}(V)$ of fields on a linear space V satisfying:

(i) any two fields in $\mathfrak{A}$ are local.

(ii) $\mathfrak{A}$ is closed under the normal products, i.e. $\mathfrak{A} \circ_n \mathfrak{A} \subset \mathfrak{A}$ for any $n \in \mathbb{Z}$.

(iii) $\mathfrak{A}$ contains the vacuum element, i.e. $\mathbb{1}_V(z) \in \mathfrak{A}$.

$\mathbb{1}_V(z)$ is called the *vacuum element* of $\mathfrak{A}$ and denoted by $\mathbb{1}_{\mathfrak{A}}$.

By Dong's lemma, we can understand that a vertex algebra is an algebra generated by collections of mutually local fields under normal products $\circ_n$, $n \in \mathbb{Z}$.

Remark 2.5. Even though our definition of a vertex algebra seems to differ from the axiomatic notion of a vertex algebra (e.g. [FLM]), it is known that these two definitions are equivalent, see [L, K].

3 Virasoro vertex operator algebra

We give an example of vertex algebras, which will be play an important role in my talk.

Let $\mathrm{Vir} = \oplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} \vec{c}$ be the Virasoro algebra, an infinite dimensional Lie algebra defined by the following Lie brackets:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} \vec{c}, \quad [\vec{c}, \mathrm{Vir}] = 0. \tag{3.1}$$

Let M be a Vir-module and assume that for any $v \in M$, there exists $N \in \mathbb{Z}$ such that $L_n v = 0$ for all $n \geq N$. Moreover, we also assume that the center $\vec{c}$ acts on M by a scalar $c \in \mathbb{C}$ (the number c is called the *central charge* of the representation). Then the generating series $\omega(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \in \mathrm{End}(M)[\![z, z^{-1}]\!]$ defines a field on M. By the

Lie bracket structure in (3.1), one can show that $\omega(z)$ is local to itself with the order of locality 4:

$$(z_1 - z_2)^4 \omega(z_1)\omega(z_2) = (z_1 - z_2)^4 \omega(z_2)\omega(z_1).$$

Therefore, the field $\omega(z)$ together with the vacuum field $\mathbb{1}_M(z)$ on M generates a vertex algebra in $\mathcal{F}(M)$. This vertex algebra is referred to as a *Virasoro vertex operator algebra* and denoted by $\mathrm{Vir}_c(\omega(z))$. It is known that $\mathrm{Vir}_c(\omega(z))$ possesses a unique simple quotient (cf. [K]). In my talk, the simple Virasoro vertex operator algebra with central charge $c = 1/2$ will be used to define involutions of a vertex operator algebra.

Since we have no more space to continue, I end this article here but now I expect the reader has studied enough fundamentals of a vertex operator algebra. So we refer the reader to the introduction of [LYY1] to get a perspective of our works on McKay's E_8-observation problem.

References

[ATLAS] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *ATLAS of Finite Groups*, Oxford Univ. Press, 1985.

[B] R.E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[C] J.H. Conway, A simple construction for the Fisher-Griess Monster group, *Invent. Math.* **79** (1985), 513–540.

[CN] J.H. Conway, and S.P. Norton, Monstrous Moonshine, *Bull. London Math. Soc.* **11**, 308–339, 1979.

[FLM] I.B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. **134**, Academic Press, New York, 1988.

[K] V.G. Kac, Vertex algebras for beginners, second edition, Cambridge University Press, Cambridge, 1990.

[LYY1] C.H. Lam, H. Yamada and H. Yamauchi, Vertex operator algebras, extended E_8 diagram, and McKay's observation on the Monster simple group, to appear in *Trans. Amer. Math. Soc.*; see also `arXiv:math.QA/0403010`.

[LYY2] C.H. Lam, H.Yamada and H. Yamauchi, McKay's observation and vertex operator algebras generated by two conformal vectors of central charge 1/2, *Internat. Math. Res. Papers* **3** (2005), 117–181.

[L] H. Li, Local systems of vertex operators, vertex superalgebras and modules, *J. Pure Appl. Algebra* **109** (1996), 143–195.

[Mc] J. McKay, Graphs, singularities, and finite groups, *Proc. Symp. Pure Math.*, Vol. **37**, Amer. Math. Soc., Providence, RI, 1980, pp. 183–186.