Sobolev's imbedding theorem in the limiting case with Lorentz space and BMO

Hideo Kozono, Kouei Minamidate, Hidemitsu Wadade

Mathematical Institute, Tohoku University, 980-8578, Sendai, Japan

We consider the Gagliardo-Nirenberg type inequality in $\mathbb{R}^n$. Let Ω be an arbitrary domain in $\mathbb{R}^n$. It is well known that the Sobolev space $H_0^{n/p,p}(\Omega), 1 < p < \infty$, is continuously embedded into $L^q(\Omega)$ for all q with $p \leqq q < \infty$. However, we cannot take $q = \infty$ in such an embedding. When $\Omega = \mathbb{R}^n$, Ogawa [11] and Ogawa-Ozawa [12] treated the Hilbert space $H^{n/2,2}(\mathbb{R}^n)$ and then Ozawa [15] gave the following general embedding theorem in the Sobolev space $H^{n/p,p}(\mathbb{R}^n)$ of the fractional derivatives which states that

$$\|\Phi_p(\alpha|u|^{p'})\|_{L^1(\mathbb{R}^n)} \leqq C\|u\|_{L^p(\mathbb{R}^n)}^p \tag{0.1}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leqq 1$, where

$$\Phi_p(\xi) := \exp(\xi) - \sum_{j=0}^{j_p-1} \frac{\xi^j}{j!} = \sum_{j=j_p}^{\infty} \frac{\xi^j}{j!}, \quad j_p := \min\{j \in \mathbb{N} \mid j \geqq p - 1\}.$$

The advantage of (0.1) gives the scale invariant form. In order to prove the above Trudinger type inequality, Ozawa [15] showed the following Gagliardo-Nirenberg type interpolation inequality which is equivalent to (0.1). For $1 < p < \infty$, there is a constant M depending only on n and p such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq Mq^{1/p'}\|u\|_{L^p(\mathbb{R}^n)}^{p/q}\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \tag{0.2}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all q with $p \leqq q < \infty$. Our goals are the generalizations of (0.2) to the Gagliardo-Nirenberg type interpolation inequality with the Lorentz space and BMO. We shall state main theorems below.

Theorem 0.1. *Let* $1 < p_1 < \infty$.
(i) *There exists a constant* $C_{n,\,p_1}$ *depending only on* n *and* p_1 *such that*

$$\|u\|_{L^q} \leqq C_{n,\,p_1}\, q^{1/r_2'}\|u\|_{L^{(p_1,p_2)}}^{p_1/q}\|(-\Delta)^{n/(2r_1)}u\|_{L^{(r_1,r_2)}}^{1-p_1/q} \tag{0.3}$$

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holds for all $u \in L(p_1, p_2)$ with $(-\Delta)^{n/(2r_1)}u \in L(r_1, r_2)$, where p_2, q, r_1 and r_2 are any numbers satisfying $1 \leqq p_2 \leqq p_1 \leqq q < \infty$, $p_1 \leqq r_1 < \infty$ and $1 < r_2 < \infty$.

(ii) There exists a constant $C_{n,\,p_1}$ depending only on n and p_1 such that

$$\|u\|_{L^q} \leqq C_{n,\,p_1}\, \frac{q^2}{q - p_1} \|u\|_{L(p_1,\infty)}^{p_1/q} \|(-\Delta)^{n/(2r_1)}u\|_{L(r_1,\infty)}^{1-p_1/q} \tag{0.4}$$

holds for all $u \in L(p_1, \infty)$ with $(I - \Delta)^{n/(2r_1)}u \in L(r_1, \infty)$, where q and r_1 are any numbers satisfying $p_1 < q < \infty$ and $p_1 \leqq r_1 < \infty$.

We note that when we put $p_1 = p_2 = r_1 = r_2 =: p \in (1, \infty)$ in (0.3), we can obtain (0.2) proved by Ozawa [15] immediately. Moreover, from Corollary 0.1, we obtain the Trudinger type inequalities equivalent to (0.3) and (0.4) as follows :

Corollary 0.1. *Let* $1 < p_1 < \infty$.
(i) *For every* $1 < r_2 < \infty$, *there exists a constant* $C_{n,\,p_1,\,r_2}$ *depending only on* n, p_1 *and* r_2 *such that the following holds. For arbitrary* $0 < \alpha < C_{n,\,p_1,\,r_2}$, *there exists a constant* $\tilde{C}_{n,\,p_1,\,r_2,\,\alpha}$ *depending only on* n, p_1, r_2 *and* α *such that*

$$\int_{\mathbb{R}^n} \Phi_{p_1,\,r_2}\left(\alpha\left(\frac{|u(x)|}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1,r_2)}}\right)^{r_2'}\right) dx \leqq \tilde{C}_{n,\,p_1,\,r_2,\,\alpha}\left(\frac{\|u\|_{L(p_1,p_2)}}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1,r_2)}}\right)^{p_1} \tag{0.5}$$

holds for all $u \in L(p_1, p_2) \setminus \{0\}$ with $(-\Delta)^{n/(2r_1)}u \in L(r_1, r_2)$, where p_2, r_1 are any numbers satisfying $1 \leqq p_2 \leqq p_1$, $p_1 \leqq r_1 < \infty$ and $\Phi_{p_1,\,r_2}$ is defined by

$$\Phi_{p_1,\,r_2}(\xi) := \sum_{\substack{r_2' j \geqq p_1 \\ j \in \mathbb{N}}} \frac{\xi^j}{j!} \qquad for \ \xi \in \mathbb{R}.$$

(ii) *There exists a constant* $C_{n,\,p_1}$ *depending only on* n *and* p_1 *such that the following holds. For arbitrary* $0 < \alpha < C_{n,\,p_1}$, *there exists a constant* $\tilde{C}_{n,\,p_1,\,\alpha}$ *depending only on* n, p_1 *and* α *such that*

$$\int_{\mathbb{R}^n} \tilde{\Phi}_{p_1}\left(\alpha\, \frac{|u(x)|}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1,\infty)}}\right) dx \leqq \tilde{C}_{n,\,p_1,\,\alpha}\left(\frac{\|u\|_{L(p_1,\infty)}}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1,\infty)}}\right)^{p_1} \tag{0.6}$$

holds for all $u \in L(p_1, \infty) \setminus \{0\}$ with $(I - \Delta)^{n/(2r_1)}u \in L(r_1, \infty)$ and for all $p_1 \leqq r_1 < \infty$, where $\tilde{\Phi}_{p_1}$ is defined by

$$\tilde{\Phi}_{p_1}(\xi) := \sum_{\substack{j > p_1 \\ j \in \mathbb{N}}} \frac{\xi^j}{j!} \qquad for \ \xi \in \mathbb{R}.$$

In fact, noting the definition of $\Phi_{p_1, r_2}($ or $\tilde{\Phi}_{p_1})$, we exchange the integral of (0.5)(or (0.6)) for the sum, and then by applying (0.3)(or (0.4)) for each integral, we have the Trudinger type inequality.

By putting $q_1 = q_2$ in Theorem 0.2, we have the following Corollary 0.2.

Theorem 0.2. (i) *For every* $1 \leqq p_1 < \infty$, *there exists a constant* C_{n, p_1} *depending only on* n *and* p_1 *such that*

$$\|u\|_{L^q} \leqq C_{n, p_1} \, q \, \|u\|_{L(p_1, p_2)}^{p_1/q} \|u\|_{BMO}^{1 - p_1/q} \tag{0.7}$$

holds for all $u \in L(p_1, p_2) \cap BMO$, *where* p_2 *and* q *are any numbers satisfying* $1 \leqq p_2 \leqq p_1 \leqq q < \infty$.

(ii) *For every* $1 < p_1 < \infty$, *there exists a constant* C_{n, p_1} *depending only on* n *and* p_1 *such that*

$$\|u\|_{L^q} \leqq C_{n, p_1} \frac{q^2}{q - p_1} \|u\|_{L(p_1, \infty)}^{p_1/q} \|u\|_{BMO}^{1 - p_1/q} \tag{0.8}$$

holds for all $u \in L(p_1, \infty) \cap BMO$ *and for all* $p_1 < q < \infty$.

Moreover, from Corollary 0.2, we obtain the Trudinger type inequalities equivalent to (0.7) and (0.8) as follows:

Corollary 0.2. (i) *For every* $1 \leqq p_1 < \infty$, *there exists a constant* C_{n, p_1} *depending only on* n *and* p_1 *such that the following holds. For arbitrary* $0 < \alpha < C_{n, p_1}$, *there exists a constant* $\tilde{C}_{n, p_1, \alpha}$ *depending only on* n, p_1 *and* α *such that*

$$\int_{\mathbb{R}^n} \Phi_{p_1}\left(\alpha \, \frac{|u(x)|}{\|u\|_{BMO}} \right) dx \leqq \tilde{C}_{n, p_1, \alpha} \left(\frac{\|u\|_{L(p_1, p_2)}}{\|u\|_{BMO}} \right)^{p_1}$$

holds for all $u \in L(p_1, p_2) \cap BMO \setminus \{0\}$ *and for all* $1 \leqq p_2 \leqq p_1$, *where* Φ_{p_1} *is defined by*

$$\Phi_{p_1}(\xi) := \sum_{\substack{j \geqq p_1 \\ j \in \mathbb{N}}} \frac{\xi^j}{j!} \qquad \text{for } \xi \in \mathbb{R}.$$

(ii) *For every* $1 < p_1 < \infty$, *there exists a constant* C_{n, p_1} *depending only on* n *and* p_1 *such that the following holds. For arbitrary* $0 < \alpha < C_{n, p_1}$, *there exists a constant* $\tilde{C}_{n, p_1, \alpha}$ *depending only on* n, p_1 *and* α *such that*

$$\int_{\mathbb{R}^n} \tilde{\Phi}_{p_1}\left(\alpha \, \frac{|u(x)|}{\|u\|_{BMO}} \right) dx \leqq \tilde{C}_{n, p_1, \alpha} \left(\frac{\|u\|_{L(p_1, \infty)}}{\|u\|_{BMO}} \right)^{p_1}$$

holds for all $u \in L(p_1, \infty) \cap BMO \setminus \{0\}$, *where* $\tilde{\Phi}_{p_1}$ *is defined by*

$$\tilde{\Phi}_{p_1}(\xi) := \sum_{\substack{j > p_1 \\ j \in \mathbb{N}}} \frac{\xi^j}{j!} \qquad \text{for } \xi \in \mathbb{R}.$$

Finally, we shall state the application to the Brezis-Gallouet-Wainger type inequality. In fact, from Corollary 0.1 (i), we can obtain the inequality as follows :

Theorem 0.3. *For every $1 < p_1 < \infty$, $1 \leqq q \leqq \infty$ and $n/q < m < \infty$, there exists a constant $C_{n,p_1,q,m}$ depending only on n, p_1, q and m such that*

$$\|u\|_{L^\infty} \leqq C_{n,p_1,q,m} \Big[1 + \big(\|u\|_{L(p_1,p_2)} + \|(-\Delta)^{n/(2r_1)}u\|_{L(r_1,r_2)} \big)$$
$$\times \big(\log(e + \|(-\Delta)^{m/2}u\|_{L^q}) \big)^{1/r'_2} \Big] \qquad (0.9)$$

holds for all $u \in L(p_1,p_2)$ with $(-\Delta)^{n/(2r_1)}u \in L(r_1,r_2)$ and $(-\Delta)^{m/2}u \in L^q$, where p_2, r_1 and r_2 are any numbers satisfying $1 \leqq p_2 \leqq p_1 \leqq r_1 < \infty$ and $1 \leqq r_2 < \infty$.

we note that the inequality with $p_1 = p_2 = r_1 = r_2 =: p \in (1,\infty)$ in (0.9) coincides with the classical Brezis-Gallouet-Wainger inequality.

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