

On the univalence superselection rule and characterization of spontaneous symmetry breaking

Hajime Moriya

Abstract

Thermodynamical formulations appropriate for general quasi-local systems with any statistics, such as characterizations of equilibrium states and spontaneous symmetry breaking are presented. We introduce a criterion of spontaneously symmetry breaking in terms of the given quasi-local structure: Each pair of distinct phases appeared in spontaneous symmetry breaking should be disjoint not only for the total system but also for every complementary outside system of a local region, which is a stronger requirement than the usual one. We derive the absence of SSB in the above sense for fermion grading transformations that multiply fermion fields by -1 . This result can be considered as a rigorous justification of the univalence super-selection rule.

We have the following result on equilibrium states exclusively for lattice systems: the violation of univalence superselection rule for even dynamics at non-zero temperature induces the non-equivalence of the KMS condition and our local thermodynamical stability condition (a kind of minimum free energy condition as open systems).

1 Introduction

It is believed that nature is subject to several super-selection rules. Among them, the univalence superselection rule claims that the superposition of two states whose total angular momenta are integers and half-integers does not exist [4] [2]. But if the number of degrees of freedom is infinite as usually considered in statistical mechanics and quantum field theory, then one may wonder whether fermion grading symmetry assumed for kinematics always leads its preservation in the state level, i.e. the absence of spontaneously symmetry breaking. In fact there is a known example of symmetry breakdown for fermion grading, though being very technical [3]. The purpose of this study is to justify the univalence super-selection as a rigid rule by proving under a model independent setting that the breakdown of fermion

grading symmetry never happens in such a way that it can be observed *macroscopically*.

It is a well known fact that if a state is invariant under some asymptotically abelian group of automorphisms like space-translations, then fermion grading symmetry is *perfectly* unbroken, that is, any such invariant state has zero expectations for every odd element (see Exam. 5.2.21 of [1]). We discuss the status of fermion grading symmetry for general fermion or fermion-boson systems without such translation invariant assumption. Doplicher-Haag-Roberts [2] theory rigorously derives the Boson-Fermion alternative. But it is a theory about Minkowski space, and gives no information for lattice models.

2 Notation

We recall the definition of quasi-local \mathbf{C}^* -systems. (We refer e.g. to § 2.6 of [1].) Let \mathfrak{F} be a directed set with a partial order relation \geq and an orthogonal relation \perp satisfying the following conditions:

- a) If $\alpha \leq \beta$ and $\beta \perp \gamma$, then $\alpha \perp \gamma$.
- b) For each $\alpha, \beta \in \mathfrak{F}$, there exists a unique upper bound $\alpha \vee \beta \in \mathfrak{F}$ which satisfies $\gamma \geq \alpha \vee \beta$ for any $\gamma \in \mathfrak{F}$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$.
- c) For each $\alpha \in \mathfrak{F}$, there exists a unique α_c in \mathfrak{F} satisfying $\alpha_c \perp \alpha$ and $\alpha_c \geq \beta$ for any $\beta \in \mathfrak{F}$ such that $\beta \perp \alpha$.

We consider a \mathbf{C}^* -algebra \mathcal{A} furnished with the following structure. Let $\{\mathcal{A}_\alpha; \alpha \in \mathfrak{F}\}$ be a family of \mathbf{C}^* -subalgebras of \mathcal{A} with the index set \mathfrak{F} . Let Θ be an involutive $*$ -automorphism of \mathcal{A} that determines the grading on \mathcal{A} as

$$\mathcal{A}^e := \{A \in \mathcal{A} \mid \Theta(A) = A\}, \quad \mathcal{A}^o := \{A \in \mathcal{A} \mid \Theta(A) = -A\}. \quad (1)$$

These \mathcal{A}^e and \mathcal{A}^o are called the even and the odd parts of \mathcal{A} . For $\alpha \in \mathfrak{F}$

$$\mathcal{A}_\alpha^e := \mathcal{A}^e \cap \mathcal{A}_\alpha, \quad \mathcal{A}_\alpha^o := \mathcal{A}^o \cap \mathcal{A}_\alpha. \quad (2)$$

The above grading structure is referred to as fermion grading (see L4 below). For a given state ω on \mathcal{A} , its restriction to \mathcal{A}_α is denoted ω_α . If a state takes zero on all odd elements, it is called an even state.

Let $\mathfrak{F}_{\text{loc}}$ be a subset of \mathfrak{F} corresponding to the set of indices of all local subsystems and set $\mathcal{A}_{\text{loc}} := \bigcup_{\alpha \in \mathfrak{F}_{\text{loc}}} \mathcal{A}_\alpha$. We assume L1, L2, L3, L4 as follows:

- L1. $\mathcal{A}_{\text{loc}} \cap \mathcal{A}_\delta$ is norm-dense in \mathcal{A}_δ for any $\delta \in \mathfrak{F}$.
- L2. If $\alpha \geq \beta$, then $\mathcal{A}_\alpha \supset \mathcal{A}_\beta$.
- L3. $\Theta(\mathcal{A}_\alpha) = \mathcal{A}_\alpha$ for all $\alpha \in \mathfrak{F}$.
- L4. For $\alpha \perp \beta$ the following graded commutation relations hold

$$\begin{aligned} [\mathcal{A}_\alpha^e, \mathcal{A}_\beta^e] &= 0, & [\mathcal{A}_\alpha^e, \mathcal{A}_\beta^o] &= [\mathcal{A}_\alpha^o, \mathcal{A}_\beta^e] = 0, \\ \{\mathcal{A}_\alpha^o, \mathcal{A}_\beta^o\} &= 0, \end{aligned}$$

where $[A, B] = AB - BA$ is the commutator and $\{A, B\} = AB + BA$ is the anti-commutator.

3 A criterion of spontaneous symmetry break-down appropriate for general quasi-local systems and fermion grading symmetry

A pair of states are called disjoint with each other if their GNS representations are disjoint. We shall employ the following more demanding condition for disjointness of two states.

Definition 1. *Let ω_1 and ω_2 be states of a quasi-local system $(\mathcal{A}, \{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\text{loc}}})$. If for every $\gamma \in \mathfrak{F}_{\text{loc}}$, their restrictions to the complementary outside system of γ , i.e., $\omega_{1\gamma_c}$ and $\omega_{2\gamma_c}$ are disjoint with each other, then ω_1 and ω_2 are said to be disjoint with respect to the quasi-local structure.*

We define a criterion of spontaneously symmetry breaking based on Definition 1 as follows. Let G be a group and $\tau_g (g \in G)$ be its action of *-automorphisms on a quasi-local system $(\mathcal{A}, \{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\text{loc}}})$. Suppose that τ_g commutes with a given (Hamiltonian) dynamics for every $g \in G$. Let Λ denote some set of physical states (e.g. the set of all ground states or all equilibrium states at some temperature for the given dynamics), and Λ^G denote the set of all G -invariant states in Λ . Let ω be an extremal point in Λ^G . Suppose that ω has a factor state decomposition in Λ in the form of $\omega = \int d\mu(g)\omega_g$ with $\omega_g := \tau_g^*\omega_0 (= \omega_0 \circ \tau_g)$, where ω_0 is a factor state in Λ (but not in Λ^G) and so is each ω_g , and μ denotes some probability measure on G . With this setting, we define the following.

Definition 2. *If for each $g \neq g'$ of G a pair of factor states ω_g and $\omega_{g'}$ are disjoint with respect to the given quasi-local structure, then it is said that the G -symmetry is macroscopically broken.*

The following proposition asserts that fermion grading symmetry cannot be broken macroscopically. A remarkable thing is that it makes no reference to the dynamics. We are using essentially no more than the canonical anticommutation relations (CAR) for its proof (see our original article).

Proposition 1. *Let ω be a state of a quasi-local system $(\mathcal{A}, \{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\text{loc}}})$ and Θ denote the fermion grading involution of \mathcal{A} . Suppose that ω is a factor state. Then ω and $\omega\Theta$ cannot be disjoint in the sense of Definition 1.*

We shall discuss more detailed results for lattice systems in the conference emphasizing on difference between spin lattice systems and fermion systems. We provide summary in the next section.

4 Fermion grading symmetry for equilibrium states of lattice systems

Take \mathbb{Z}^ν , ν -dimensional cubic integer lattice. Let $\mathfrak{F}_{\text{loc}}$ be a set of all finite subsets of the lattice. We assume that there is a finite number of degrees of freedom (spins) on each site of the lattice. We further assume the uniformity, i.e. the subalgebra $\mathcal{A}_{\{i\}}$ on each site i on the lattice is isomorphic to a $d \times d$, full matrix algebra, $d \in \mathbb{N}$ being independent of i . On each site $i \in \mathbb{Z}^\nu$, $\mathcal{A}_{\{i\}}$ is generated by fermion operators a_i, a_i^* , and spin operators represented by the Pauli matrices $\sigma_i^x, \sigma_i^y, \sigma_i^z$ which are even elements commuting with all fermion operators.)

Let α_t ($t \in \mathbb{R}$) be a one-parameter group of $*$ -automorphisms of \mathcal{A} . A state φ is called an (α_t, β) -KMS state if it satisfies

$$\varphi(A\alpha_{i\beta}(B)) = \varphi(BA)$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{A}_{\text{ent}}$, where \mathcal{A}_{ent} denotes the set of all $B \in \mathcal{A}$ for which $\alpha_t(B)$ has an analytic extension to \mathcal{A} -valued entire function $\alpha_z(B)$ as a function of $z \in \mathbb{C}$.

Our dynamics α_t is assumed to be even, namely $\alpha_t \Theta = \Theta \alpha_t$. We also assume the following (I, II) in order to relate α_t with some $\delta \in D(\mathcal{A}_{\text{loc}})$.

- (I) The domain of the generator δ_α of α_t includes \mathcal{A}_{loc} .
- (II) \mathcal{A}_{loc} is a core of δ_α .

The next statement asserts that for even dynamics α_t any even KMS cannot be decomposed into non-even KMS states which are locally thermodynamical stable, called LTS (a kind of variational principle, minimum free energy condition for open systems).

Proposition 2. *Let α_t be an even dynamics satisfying (I, II) and let φ be an arbitrary even (α_t, β) -KMS state. For $I \in \mathfrak{F}_{\text{loc}}$, let $\tilde{\alpha}_t$ denote the perturbed dynamics of the given α_t by the local Hamiltonian $H(I)$. Let Φ be the potential induced by α_t and $\tilde{\Phi}$ be that for $\tilde{\alpha}_t$, which is concretely given as*

$$\tilde{\Phi}(J) := 0, \text{ if } J \cap I \neq \emptyset, \quad \text{and} \quad \tilde{\Phi}(J) := \Phi(J), \text{ otherwise.} \quad (3)$$

If the odd part of the center of φ , equivalently that of $\varphi^{\beta H(I)}$, is non empty, then the induced noneven $(\tilde{\alpha}_t, \beta)$ -KMS states ψ and $\psi\Theta$ violate $(\tilde{\Phi}, \beta)$ -LTS condition.

5 Conclusions

We have shown that the univalence superselection rule is always satisfied in our formulation of SSB that respects a given quasi-local structure. We have shown that if there are noneven KMS states, then they inevitably invalidate the local thermal stability condition. It can be said that fermion grading symmetry breaking, if it would occur, is pathological from a thermodynamical viewpoint.

References

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