

A new example of supergroups

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1 Introduction

The ground field is fixed to \mathbb{C} in this talk.

The notion of supergroups arises in the middle of 1960's in the study of the Hamiltonian formalism of spin systems ([1]), and it is applied for supersymmetric (SUSY) quantum field theory, especially supergravity (SUGRA).

There is many applications of supergroups for physics, though we restrict our interest to purely mathematical aspects in this talk.

Supergroups are groups and they are roughly separated to two classes, algebraic supergroups and Lie supergroups. We can consider that algebraic supergroups are a type of generalization of algebraic groups, and Lie supergroups are a type of generalization of Lie groups. Many methods to study algebraic groups or Lie groups is available for also supergroups. A Lie supergroup $D(2, 1; i)$ ($i = \sqrt{-1}$) is dealt in this talk.

In this section, we see where this $D(2, 1; i)$ stands in whole theory of Lie supergroups. Like as Lie groups and Lie algebras, we can consider Lie correspondence. For each Lie supergroups, the corresponding infinitesimal object is called its Lie superalgebra.

The classification of finite-dimensional complex simple Lie superalgebras is accomplished by Kac[4], and he roughly classified them to four types, classical, exceptional, Cartan type, and queer type.

$D(2, 1; a)$ is a family of exceptional Lie superalgebras parametrized $a \in \mathbb{C}$, and a Lie supergroups corresponding to $D(2, 1; a)$ is the topics of this talk.

A maximal solvable subgroup is called a Borel subuspergroup. The purpose of this talk is to determine the exponential mapping of a Borel subuspergroup of G such that $\text{Lie}(G) = D(2, 1; i)$.

2 Lie superalgebra $D(2, 1; a)$

The definition of Lie superalgebras is the following

Definition 1 Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. We denote the grading of $v \in \mathfrak{g}_0 \amalg \mathfrak{g}_1$ by \tilde{v} . In other words, $\tilde{v} = 0$ if and only if $v \in \mathfrak{g}_0$, and $\tilde{v} = 1$ if and only if $v \in \mathfrak{g}_1$. We should remark $\tilde{v} \in \mathbb{Z}/2\mathbb{Z}$. If a bilinear form $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfy three following conditions

(1)

$$\widetilde{[u, v]} = \tilde{u} + \tilde{v}.$$

(2)

$$[u, v] = -(-1)^{\tilde{u}\tilde{v}}[v, u].$$

(3)

$$[u, [v, w]] + (-1)^{(\tilde{u}+\tilde{v})\tilde{w}}[w, [u, v]] + (-1)^{\tilde{u}(\tilde{v}+\tilde{w})}[v, [w, u]] = 0,$$

$[\cdot, \cdot]$ is called a Lie superbracket and \mathfrak{g} is called a Lie superalgebra.

The notion of roots has meanings also in the theory of Lie superalgebras. And a simple Lie superalgebra can be identified with Dynkin diagrams and Cartan matrices. The differenc point is the correspondence between Lie superalgebras and Dynkin diagrams (and also Cartan matrices) is not generally 1 to 1.

A standard Cartan matrix of $D(2, 1; a)$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -a \\ 0 & -a & 2a \end{pmatrix}. \quad (1)$$

The construction of the simple Lie superalgebra whose Cartan matrix is (1) is the following. Prepare generators $h_1, h_2, h_3, e_1, e_2, e_3, f_1, f_2, f_3$ (only e_2 and f_2 are odd elements) and assume a relation

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ij}e_j \\ [h_i, f_j] &= a_{ij}f_j \\ [e_i, f_j] &= \delta_{ij}h_j \end{aligned}$$

where a_{ij} are elements of Cartan matrix and δ_{ij} is the Kronecker delta. Then a vector space spanned by

$$\begin{aligned} &\{h_i, h_j, h_k; 1 \leq i, j, k \leq 3\} \cup \\ &\{[e_{i_1}, [e_{i_2}, [\cdots, [e_{i_{k-1}}, e_{i_k}]] \cdots]]; 1 \leq i_1, \cdots, i_k \leq 3, k \in \mathbb{N}\} \cup \\ &\{[f_{i_1}, [f_{i_2}, [\cdots, [f_{i_{l-1}}, f_{i_l}]] \cdots]]; 1 \leq i_1, \cdots, i_l \leq 3, l \in \mathbb{N}\} \end{aligned}$$

is a Lie superalgebra, and we denote it by $D(2, 1; a)$. We denote

$$\begin{aligned} \mathfrak{h} &:= \langle h_1, h_2, h_3 \rangle \\ \tilde{\mathfrak{n}}_+ &:= \langle e_1, e_2, e_3, [e_1, e_2], \cdots, [e_{i_1}, [e_{i_2}, [\cdots, [e_{i_{k-1}}, e_{i_k}]] \cdots]], \cdots \rangle \\ \tilde{\mathfrak{n}}_- &:= \langle f_1, f_2, f_3, [f_1, f_2], \cdots, [f_{i_1}, [f_{i_2}, [\cdots, [f_{i_{l-1}}, f_{i_l}]] \cdots]], \cdots \rangle \end{aligned}$$

$\widetilde{D(2, 1; a)}$ is not a simple Lie superalgebra, and we need to take the quotient algebra by a maximal ideal contained in $\tilde{\mathfrak{n}}_+ \cup \tilde{\mathfrak{n}}_-$ in order to construct a simple Lie superalgebra. A maximal ideal like such is

$$\begin{aligned} J = & \langle [e_1, e_3], [e_2, e_2], [e_1, [e_1, e_2]], [e_2, [e_1, e_2]], [e_2, [e_2, e_3]], [e_3, [e_2, e_3]], \\ & [e_1, [e_1, [e_2, e_3]]], [e_3, [e_1, [e_2, e_3]]], [e_1, [e_2, [e_1, [e_2, e_3]]]], [e_3, [e_2, [e_1, [e_2, e_3]]]] \rangle \\ & \oplus \langle [f_1, f_3], [f_2, f_2], [f_1, [f_1, f_2]], [f_2, [f_1, f_2]], [f_2, [f_2, f_3]], [f_3, [f_2, f_3]], \\ & [f_1, [f_1, [f_2, f_3]]], [f_3, [f_1, [f_2, f_3]]], [f_1, [f_2, [f_1, [f_2, f_3]]]], [f_3, [f_2, [f_1, [f_2, f_3]]]] \rangle \end{aligned}$$

We define

$$D(2, 1; a) := \widetilde{D(2, 1; a)} / J, \quad \mathfrak{n}_+ := \tilde{\mathfrak{n}}_+ / (J \cap \tilde{\mathfrak{n}}_+), \quad \mathfrak{n}_- := \tilde{\mathfrak{n}}_- / (J \cap \tilde{\mathfrak{n}}_-)$$

and

$$\begin{aligned} e_4 & := [e_1, e_2], & e_5 & := [e_2, e_3], & e_6 & := [e_3, [e_2, e_1]], & e_7 & := [e_2, [e_3, [e_2, e_1]]] \\ f_4 & := [f_1, f_2], & f_5 & := [f_2, f_3], & f_6 & := [f_3, [f_2, f_1]], & f_7 & := [f_2, [f_3, [f_2, f_1]]]. \end{aligned}$$

$\{e_1, \dots, e_7\}$ form a basis of \mathfrak{n}_+ and $\{f_1, \dots, f_7\}$ form a basis of \mathfrak{n}_- , so $D(2, 1; a)$ is a 9|8-dimensional Lie superalgebra.

3 An irreducible representation of $D(2, 1; i)$

In order to construct a finite dimensional irreducible representation of $D(2, 1; i)$, we use a type of infinite dimensional representation— called Verma module, and its quotient module. The construction of Verma module does not depend on $a \in \mathbb{C}$, though we construct an irreducible representation in the situation a is fixed to i . For this reason, we deal with the case of general $a \in \mathbb{C}$ in the first half of this section, and deal with the case of $a = i$ in the latter half of this section.

Like as Lie algebras, we can consider objects like universal enveloping algebras, and it is called universal enveloping superalgebras ([5]). For Lie superalgebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (Gel'fand–Naimark decomposition), its Verma module M_Λ depends on $\Lambda \in \mathfrak{h}^*$. Its representation space is defined by

$$M_\Lambda := \mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{C}v_\Lambda$$

(v_Λ is a non-zero vector of 1-dimensional vector space $\mathbb{C}v_\Lambda$), and the action of \mathfrak{g} on M_Λ is defined by

$$\begin{aligned} h_i(1 \otimes v_\Lambda) & := \Lambda(h_i) \otimes v_\Lambda \\ e_i(1 \otimes v_\Lambda) & := 0 \\ f_i(1 \otimes v_\Lambda) & := f_i \otimes v_\Lambda \end{aligned}$$

There is a nice class in Lie superalgebras called generalized Kac–Moody superalgebras, and representation theory on it is well-studied. On the other

hand, $D(2, 1; a)$ is not a generalized Kac–Moody superalgebra if $a \neq 1$, so we cannot apply this theory directly to $D(2, 1; i)$. However, we can get the following results in the $D(2, 1; i)$ case.

Proposition 1 *If $\mathfrak{g} = D(2, 1; i)$ and $\Lambda(h_3) = 0$,*

(1)

$$U_0 := \mathcal{U}(\mathfrak{n}_-) f_1^{\Lambda(h_1)+1} \oplus \mathcal{U}(\mathfrak{n}_-) f_3$$

is a submodule of M_Λ .

(2) *If a submodule U of M_Λ contains U_0 , then the quotient module M_Λ/U is an integrable representation.*

Proposition 2 *If $\Lambda(h_1) = 1, \Lambda(h_2) = 0, \Lambda(h_3) = 0$,*

(1) *There are 8 different integrable representations and 7 of them are reducible.*

(2)

$$U := \langle f_2, f_3, f_1 f_2 f_1, f_1 f_3 f_2 f_1, f_3^2 f_2 f_1, f_3 f_2 f_3 f_2 f_1, f_1^2 f_2 f_3 f_2 f_1, \\ f_3 f_1 f_2 f_3 f_2 f_1, f_3 f_2 f_1 f_2 f_3 f_2 f_1, f_1 f_2 f_1 f_2 f_3 f_2 f_1 \rangle$$

is the maximal proper submodule such that $U \supset U_0$ and M_Λ/U is a 4|3-dimensional irreducible representation.

4 The exponential of a Borel subsuperalgebra of $D(2, 1; i)$

Roughly speaking, Lie supergroups are supermanifolds with groups structure and we regard them as a type of generalization of Lie groups. See [2] for general theory of supermanifolds and Lie supergroups. Like as Lie groups and Lie algebras, there is Lie correspondence between Lie supergroups and Lie superalgebras, and it is given by the exponential mapping.

We have found 4|3-dimensional representation of $D(2, 1; i)$, and we can consider

$$\text{Exp}(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k \quad (2)$$

where $A \in \pi(D(2, 1; i))$ (Here we denote π the representation got in the previous section.). (2) will be got by solving a system of inhomogeneous linear ODEs, so it is possible that we will get the explicit form. However, the solution of the system of ODEs defining (2) is very complicated, and it is hard to calculate. For this reason, the author determined $\text{Exp}(A)$ for $A \in \mathfrak{n}_+ \oplus \mathfrak{h}$.

Berezin–Kac proved the exponential mapping is a local isomorphism in the category of supermanifolds, so determining the explicit form of the exponential mapping has the meaning as a local coordinate function near the unit element as a Lie supergroup.

References

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