A scaling limit for a general class of quantum field models and its application to nuclear physics

Akito Suzuki

Department of Mathematics, Hokkaido University

1 Introduction

We study a mathematically rigorous method of deriving the Schrödinger Hamiltonians of the form

$$H_{\mathrm{S}} = -\Delta + V, \tag{1.1}$$

where Δ denotes the generalized Laplacian. These Hamiltonians describe the total energy of systems for N nonrelativistic quantum particles removing in $\mathbb{R}^d$ under the influence of a potential V $(d, N \in \mathbb{N})$. In quantum mechanics, H_{S} is a self-adjoint operator acting on a Hilbert space. The exact nature of this Hilbert space is dependent on the potential V (hence, on the system); for example, (1) the Schrödinger Hamiltonian of particles under the influence of a scalar potential $V : \mathbb{R}^{dN} \to \mathbb{R}$ acts on the space of square-integrable functions $L^2(\mathbb{R}^{dN})$, and (2) the Schrödinger Hamiltonian of particles under the influence of an $n \times n$-matrix-valued potential acts on the space of square-inregrable $\mathbb{C}^n$-valued functions $L^2(\mathbb{R}^{dN}; \mathbb{C}^n)$. They are of general use to explain many phenomena.

A method of deriving a class of the Schrödinger Hamiltonians in the case (1) was studied by Davies [2] and Hiroshima [3, 4].

In nuclear physics and condensed matter physics, physicists often use the Schrödinger Hamiltonian in the case (2). However, as far as we know, there is little literature on a mathematically rigorous method of deriving such a Hamiltonian. We are interested in deriving the Hamiltonin in the case (2). We derive a class of the Schrödinger Hamiltonians in a scaling limit [1, 3, 7] of Hamiltonians of particles coupled to a Bose field which describes an interaction between the particles.

Scaling limit

Now, we explain the scaling limit for a model of quantum particles coupled to an abstract Bose field. To begin with, we introduce the Hamiltonian of this model by

$$H = -\Delta \otimes I + I \otimes H_{\mathrm{b}} + gH_{\mathrm{I}}, \tag{1.2}$$

acting on $L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K})$, where I denotes the identity opertor, H_{b} the free Hamiltonian of the Bose field, H_{I} a symmetric operator that describes interaction between the particles and the Bose field, g a coupling constant that denotes the strength of the interaction, and $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ the Boson Fock space over a Hilbert space $\mathcal{K}$. We detail these notations in the following section.

We introduce a scaled Hamiltonian by

$$H(\Lambda) = -\Delta \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + g\Lambda H_{\mathrm{I}}, \qquad \Lambda > 0. \tag{1.3}$$

We are interested in the limit of $H(\Lambda)$ as $\Lambda \to \infty$ in the strong resolvent sense.

In [3, 4], it was shown that, under suitable conditions,

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)-z)^{-1} = (H_{\mathrm{S}}-z)^{-1} \otimes P_0 \tag{1.4}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where P_0 denotes the orthogonal projection from $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ onto $\ker H_{\mathrm{b}}$. Note in passing that, physically, $\ker H_{\mathrm{b}}$ represents the vacuum of the Bose field. Hence, we obtain the Schrödinger Hamiltonian in the vacuum of the field. However, H_{S} in (1.4) is the Schrödinger Hamiltonian in the case (1). The purpose of this paper is to derive the Schrödinger Hamiltonian in the case (2). Indeed, we can derive the Schrödinger Hmiltonian with operator-valued potential.

2 Definition of a model

In order to derive the Schrödinger Hamiltonian in the case (2), we introduce an abstract model for quantum particles coupled to a Bose field with some degrees of freedom. We denote the one-boson Hilbert space by $\mathcal{K}$ which is taken to be an arbitrary separable complex Hilbert space. To describe the Bose field, one uses the Boson Fock space over $\mathcal{K}$:

$$\mathcal{F}_{\mathrm{b}}(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \bigotimes_{\mathrm{s}}^{n} \mathcal{K}$$

where $\bigotimes_{\mathrm{s}}^{n} \mathcal{K}$ denotes the n-fold symmetric tensor product of $\mathcal{K}$ with $\bigotimes_{\mathrm{s}}^{0} \mathcal{K} := \mathbb{C}$.

Let T be a non-negative, injective and self-adjoint operator on $\mathcal{K}$. The operator T represents a dispersion relation of one free boson associated with the Bose field under consideration. The free Hamiltonian of the Bose field is defined by the second quantization of T:

$$H_{\mathrm{b}} := d\Gamma(T)$$

acting on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$, where

$$d\Gamma(T) := \bigoplus_{n=0}^{\infty} T^{(n)},$$

with $T^{(0)} = 0$ and $T^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\overbrace{T}} \otimes \cdots \otimes I \right) \Bigg| \widehat{\bigotimes^{n}} D(T),$$

where $\bigotimes_{\mathrm{alg}}$ denotes the algebraic tensor product. Note that $d\Gamma(T)$ is non-negative, since T is non-negative.

Let $a(f)$ $(f \in \mathcal{K})$ be the annihilation operators and $a(f)^*$ the creation operators, satisfying the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{K}$ on the dense subspace

$$\mathcal{F}_0(\mathcal{K}) := \left\{\psi \in \mathcal{F}_{\mathrm{b}}(\mathcal{K}) \mid \psi^{(n)} = 0 \textit{ for any } n \geq n_0 \textit{ with some } n_0 \in \mathbb{N}\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes innerproduct of $\mathcal{K}$ and $[X, Y] := XY - YX$. The Segal field operator

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \qquad f \in \mathcal{K},$$

is essentially self-adjoint on $\mathcal{F}_0(\mathcal{K})$ [5, §X.7]. We denote its closure by the same symbol $\phi(f)$.

Let $\mathcal{H}$ be an arbitrary separable complex Hilbert space. The Hilbert space of the coupled system of the particles and the Bose field with some degrees of freedom is given by

$$\mathcal{F} := L^2(\mathbb{R}^{dN}; \mathcal{H}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K}) \simeq \mathcal{H} \otimes L^2(\mathbb{R}^{dN}; \mathcal{F}_{\mathrm{b}}(\mathcal{K})).$$

Let $B_j (j = 1, \cdots, J)$ be bounded self-adjoint operators on $\mathcal{H}$, $g_j : \mathbb{R}^{\mathbb{N}} \to \mathcal{K}$ $(j = 1, \cdots, J)$ continuous functions and

$$H_{\mathrm{I}} = \sum_{j=1}^{J} B_j \otimes \int_{\mathbb{R}^{dN}}^{\oplus} \phi(g_j(x))dx,$$

where $\int_{\mathbb{R}^{dN}}^{\oplus} \cdot dx$ denotes the operator whose fiber is $\cdot$ (see [6, §XIII.16]). We introduce a Hamiltonian of the particles coupled to the Bose field with some degrees of freedom by

$$H = -\mathit{\Delta} \otimes I + I \otimes H_{\mathrm{b}} + gH_{\mathrm{I}}$$

on $\mathcal{F}$, where $g \in \mathbb{R}$ is a coupling constant.

3 The main result

To begin with, we introduced a scaled Hamiltonian $H(\Lambda)$ $(\Lambda > 0)$ by

$$H(\Lambda) := -\mathit{\Delta} \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + g\Lambda H_{\mathrm{I}}.$$

In order to describe the main result, we now formulate hypotheses. To do this, we introduced some notations.

We denote by $L^\infty(\mathbb{R}^{dN}; \mathcal{K})$ the set of mesurable functions $f : \mathbb{R}^d \mapsto \mathcal{K}$ for which

$$\|f\|_\infty := \text{ess.} \sup_{x \in \mathbb{R}^{dN}} \|f(x)\|_{\mathcal{K}} < \infty.$$

For $\alpha \in \mathbb{R}$, we define a $\mathcal{K}$-valued function $T^\alpha f$ on $\mathbb{R}$ as follows: if $f(x) \in D(T^\alpha)$ a.e.$x \in \mathbb{R}^{dN}$ with respect to Lebesuge mesure,

$$(T^\alpha f)(x) := T^\alpha f(x).$$

Definition 3.1 *Let* $\alpha \in \mathbb{R}$. $L^\infty_\alpha(\mathbb{R}^{dN};\mathcal{K})$ *denotes the set of* $\mathcal{K}$*-valued functions* f *on* $\mathbb{R}$ *satisfing the following conditions:*
(i) f *is strongly continuous with* $f \in L^\infty(\mathbb{R}^{dN};\mathcal{K})$.
(ii) $f(x) \in D(T^\alpha)$ $(x \in \mathbb{R})$ *and* $T^\alpha f \in L^\infty(\mathbb{R}^{dN};\mathcal{K})$.

A $\mathcal{K}$-valude function f on $\mathbb{R}^{dN}$ is to be diferentiable with respect to x_μ if the sequence

$$\frac{f(x_1,\cdots,x_\mu+\varepsilon,\cdots,x_d)-f(x)}{\varepsilon} \tag{3.1}$$

converges as $\varepsilon \to 0$ for any $x=(x_1,\cdots,x_{dN}) \in \mathbb{R}^{dN}$. Then, we denote the limit of (3.1) by $\partial_\mu f$. One can define the n times diferentiability ($n \in \mathbb{N}$), inductively:

$$\partial_\mu^n f := \partial_\mu(\partial_\mu^{(n-1)} f), \ n \geq 1.$$

Hypothesis A *The functions* $g_j (j=1,\cdots,J)$ *are twice diferensiable and satisfying the following conditions:*
(i) $g_j \in L^\infty_{-3/2}(\mathbb{R}^{dN};\mathcal{K}) \cap L^\infty_{-1}(\mathbb{R}^{dN};\mathcal{K}) \cap L^\infty_{-1/2}(\mathbb{R}^{dN};\mathcal{K})$.
(ii) $\partial_\mu(T^{-1}g_j) \in L^\infty_{-1/2}(\mathbb{R}^{dN};\mathcal{K}) \cap L^\infty_{1/2}(\mathbb{R}^{dN};\mathcal{K})$ *for* $\mu=1,\cdots,d$.
(iii) $\partial_\mu^2(T^{-1}g_j) \in L^\infty_{-1/2}(\mathbb{R}^{dN};\mathcal{K})$ *for* $\mu=1,\cdots,d$.
Moreover, we assume that for any $j,k=1,\cdots,J$ *and* a.e.$x \in \mathbb{R}^{dN}$

$$\langle g_j(x), g_k(x)\rangle, \quad \langle g_j(x), T^{-1}g_k(x)\rangle, \quad \langle T^{-1}g_j(x), T^{-1}g_k(x)\rangle \in \mathbb{R}.$$

We now ready to describe our main result. Let

$$V_{\text{eff}} = -\frac{g^2}{2}\sum_{1\leq j,k\leq J} V_{j,k},$$

where

$$V_{j,k}(x) = \left\langle g_j(x), T^{-1}g_k(x)\right\rangle B_k B_j, \quad \text{a.e.}x \in \mathbb{R}^{dN}.$$

Theorem 3.2 *Assume Hypothesis A. Then, for all* $z \in \mathbb{C}\setminus\mathbb{R}$

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)-z)^{-1} = (H_{\text{eff}}-z)^{-1}\otimes P_0,$$

where

$$H_{\text{eff}} = -\Delta + V_{\text{eff}}$$

on $L^2(\mathbb{R}^{dN};\mathcal{H})$.

4 Concluding remarks

(a) Let W be a symmetric operator on $L^2(\mathbb{R}^{dN};\mathcal{H})$ obeying $-\Delta$-bounded with the relative bound less than one. Then, one can show that, under suitable conditions,

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)+W\otimes I-z)^{-1} = (H_{\text{eff}}+W)^{-1}\otimes P_0.$$

(See [7].)

(b) In this paper, ultraviolet cutoffs (see [3, 4]) remain in the scaling limit H_{eff}. It is worth studying the scaling limit with removing ultraviolet cutoffs. Indeed, one can remove an ultra violet cutoff under suitable conditions. One of the conditions is the commutativity of the operators B_j. However, generally speaking, concrete models don't satisfy this condition. We are interested in the scaling limit with removing ultraviolet cutoffs without this commutativity condition. This is an open problem.

References

[1] A. Arai, An asymptotic analysis and its application to the nonrelativistic limit of the Pauli-Fierz and a spin-boson model, *J. Math. Phys.* **31** (1990), 2653-2663.

[2] E. B. Davies, Particle-boson interactions and the weak coupling limit, *J. Math. Phys.* **20** (1979), 345-351.

[3] F. Hiroshima, Weak coupling limit with a removal of an ultraviolet cutoff for a Hamiltonian of particles interacting with a massive scalar field, *Inf. Dimen. Anal. Quantum Prob. Relat. Top.* **1** (1998), 407-423.

[4] F. Hiroshima, Weak coupling limit and removing of an ultraviolet cutoff for a Hamiltonian of particles interacting with a quantized scalar field, *J. Math. Phys.* **40** (1999), 1215-1236.

[5] M. Reed and B. Simon, *Methods of Modern Mathmatical Physics Vol. II*, Academic Press, New York (1975).

[6] M. Reed and B. Simon, *Methods of Modern Mathmatical Physics Vol. IV*, Academic Press, New York (1978).

[7] A. Suzuki, A Scaling limit for a general class of quantum field models and its applications to nuclear physics and condenced matter Physics, preprint (2005).