MREにより弾性係数と粘性係数の同定法の数理
Mathematical study of identifying of viscoelasticity by MRE

中村　玄（北大・理）　江　渝（北大・理）　永安　聖（北大・理）　程　晋（復旦・数）

Gen NAKAMURA, Hokkaido University, Japan
Yu JIANG, Hokkaido University, Japan
Sei NAGAYASU, Hokkaido University, Japan
Jin CHENG, Fudan University, China
Email: jiang@math.sci.hokudai.ac.jp

We will propose a new reconstruction scheme for identify the viscoelasticity of living body from MRE measurements. The reconstruction scheme consistes of application of oscillating - dacaying solutions, complex geometrical optics solution, solving the Cauchy problem for elliptic and hyperbolic equations.

1 Introduction

In the method of dynamic MR-Elastography, it is reasonable to consider not only the elastic properties of the material but also the viscous properties of the material. There are various models to introduce viscosity into the elastic equation. The simplest model is the so-called Voigt model.

That is, for any time $t > 0$, and a point $x = (x_1, \cdots, x_n)$ in a bounded domain $\Omega \subset \mathbb{R}^n (n = 2, 3)$ whose boundary $\partial\Omega$ is $\mathcal{C}^\infty$ smooth, the displacement $u(x, t)$ satisfies the equation:

$$\begin{cases} \rho(x)\partial_t^2 u_i - \displaystyle\sum_k \frac{\partial}{\partial_k} \sum_{lm} \lambda_{ilkm}(x) u_{lm} \\ \qquad\qquad - \displaystyle\sum_k \frac{\partial}{\partial_k} \sum_{lm} \eta_{ilkm}(x) \partial_t u_{lm} = 0, \end{cases} \tag{1.1}$$

where

$$u_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

$\rho(x) > 0$ is the density, the elasticity tensor λ_{ilkm} and the viscosity tensor η_{ilkm} satisfy the symmetries:

$$\lambda_{ilkm} = \lambda_{kilm} = \lambda_{ikml} = \lambda_{lmik},$$

$$\eta_{ilkm} = \eta_{kilm} = \eta_{ikml} = \eta_{lmik}.$$

If we assume that the material is isotropic and incompressible, the Voigt model (1.1) reduces to a scalar equation with shear modulus $\mu(x)$ and viscosity coefficient $\eta(x)$.

For simplicity, we assume $n = 2$ and $\mu(x), \eta(x), \rho(x) \in \mathcal{C}^\infty(\bar{\Omega})$ satisfy the following condition

$$\mu(x), \eta(x), \rho(x) > 0 \quad \text{on} \quad \bar{\Omega}.$$

Then, the forward problem is as follow.

Forward Problem:

For any $f \in \mathcal{C}^0([0, \infty); H^{\frac{1}{2}}(\partial\Omega))$, to find a solution $u \in \mathcal{C}^0([0, \infty); H^1(\Omega)) \cap \mathcal{C}^1([0, \infty); L^2(\Omega))$ to

$$\begin{cases} \rho(x)\partial_t^2 u - \nabla \cdot (\mu(x)\nabla u + \eta(x)\nabla u_t) = 0 \\ \qquad\qquad\qquad\qquad\qquad \text{in} \quad (0, \infty) \times \Omega, \\ u = f \qquad\qquad\qquad\quad \text{on} \quad (0, \infty) \times \partial\Omega, \\ u = u_t = 0 \qquad\qquad\quad \text{on} \quad \{0\} \times \Omega. \end{cases} \tag{1.2}$$

It is well known the forward problem is well posed. We denote the solution u to (1.2) by $u = u(f)$. Moreover, we have the following:

Proposition 1.1. *For any $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, $F \in \mathcal{C}^0([0, \infty); L^2(\Omega))$, there exists a unique solution $u \in \mathcal{C}^0([0, \infty); H^1(\Omega)) \cap \mathcal{C}^1([0, \infty); L^2(\Omega))$ to*

$$\begin{cases} \rho(x)\partial_t^2 u - \nabla \cdot (\mu(x)\nabla u + \eta(x)\nabla u_t) = F \\ \qquad\qquad\qquad\qquad\qquad \textit{in} \quad (0, \infty) \times \Omega, \\ u = 0 \qquad\qquad\qquad\quad \textit{on} \quad (0, \infty) \times \partial\Omega, \\ u = u_0, \quad u_t = u_1 \qquad \textit{on} \quad \{0\} \times \Omega. \end{cases} \tag{1.3}$$

Also, there exists a constant $c_0 > 0$ independent of u_0, u_1, F such that

$$\|u(t)\|_{H^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)} = O(e^{-c_0 t}) \quad (t \to \infty). \tag{1.4}$$

Based on the well posedness of the forward problem, we formulate our inverse problem as follow.

Inverse Problem:

Suppose $\mu(x), \eta(x), \rho(x)$ are unknown. Reconstruct $\mu(x)$, $\eta(x)$, $\rho(x)$ from $u(f)$ in $(0, T) \times \bar{\Omega}$ for finitely many f's, where $u = u(f)$ is the solution to (1.2).

Theorem 1.2. *There is a reconstruction procedure for this inverse problem.*

The details of the reconstruction procedure will be given later. In section 2, we reduce our forward problem to a kind of elliptic partial differential equation, then in section 3 we construct the oscillating-decaying solution to this elliptic equation and state an alternating iterative method to solving the Cauchy problem for this elliptic equation in section 4. Finally, in section 5, by the application of the oscillating-decaying solution to solve the first order hyperbolic equation and solve the Cauchy problem for elliptic equation repeatedly, we can reconstruct the coefficients in our inverse problem.

2 The dominant part of $u(f)$

Theorem 2.1. *Let $\omega > 0$ be a constant and $f(x.t) = e^{i\omega t}g(x)$ with $g(x) \in H^{\frac{1}{2}}(\partial\Omega)$. Then, the boundary value problem (1.2) has a unique solution $u \in \mathcal{C}^0([0,\infty); H^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega))$ with dominant part $e^{i\omega t}v(x)$ where $v(x) \in H^1(\Omega)$ solves*

$$\begin{cases} \nabla \cdot (\mu \nabla v + i\omega\eta\nabla v) + \rho\omega^2 v = 0 & \textit{in} \quad \Omega, \\ v = g & \textit{on} \quad \partial\Omega, \end{cases} \tag{2.5}$$

Remark 2.2. *Hence, we can say that we know v in Ω if we know $u(e^{i\omega t}g)$ in $\Omega \times [0,\infty)$.*

3 Application of oscillating-decaying solutions

Theorem 3.1. *By using interior measurements associated to oscillating - decaying solutions (abbreviated by OD solutions), and two different ω's, we can approximately reconstruct $\mu(x), \eta(x)$ near $\partial\Omega$ if ρ, $\mu|_{\partial\Omega}$ and $\eta|_{\partial\Omega}$ are known.*

Since the construction of the oscillating - decaying solution is local near any point on the $\partial\Omega$, Theorem 3.1 is respect to

Lemma 3.2. *By using an OD solution, and two different ω's, we can continue $\tilde{\gamma}_\omega$ from $\widetilde{\Gamma}$ into its neighborhood in the direction of y_n by solving the Cauchy problem for a hyperbolic equation with respect to $\tilde{\gamma}_\omega$.*

4 Application of complex geometrical optics solution

Base on the argument in section 3, we can smoothly extend the coefficients to an larger domain which contains Ω (see [3]). Next, in the domain $\widetilde{\Omega} \subset \mathbb{R}^n$ which is large enough (e.g. some ball B_{R_0+2}), we consider some domain Ω_1, such that $\bar{\Omega}_1 \Subset \widetilde{\Omega}$, $\mu(x), \eta(x)$ are known in $\widetilde{\Omega} \setminus \bar{\Omega}_1$ and its shape is analogous to that of $\widetilde{\Omega}$, we state an iterative method (Johansson's iteration, [1]) to solve the Cauchy problem

$$\begin{cases} Lv := \nabla \cdot (\gamma\nabla v) + \rho\omega^2 v = 0 & \text{in} \quad \widetilde{\Omega} \setminus \bar{\Omega}_1, \\ u = \varphi & \text{on} \quad \Gamma_0, \\ Nu := \partial_L u = \psi & \text{on} \quad \Gamma_0, \end{cases} \tag{4.1}$$

with an incomplete Cauchy data (i.e. the Cauchy data with noise). We would like to find the Dirichlet input g on $\partial\Omega$ such that it does not contain any unknown data and the interior measurement generated by this input approximates the CGO solution in the sense of C^k $(k \geq 2, k \in \mathbb{N})$.

5 Reconstruction of $\mu(x), \eta(x)$

By giving an specified input g, we have an interior measurement $v(g)$ which approximates to the CGO solution. By using this data we can solve the Cauchy problem for hyperbolic equation respect to γ just as we did in section 3 in the direction k repeatedly and the coefficients μ and η are reconstructed in Ω.

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