Reconstruction of cavities in medium from boundary measurements

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1 Introduction

We consider the inverse problem for identifying cavities in medium. The method is based on the enclosure method which was introduced by Ikehata [1].

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, and D be an open set in Ω with C^2 boundary satisfying $\overline{D} \subset \Omega$. We assume that $\Omega \setminus \overline{D}$ is connected and that the conductivity in $\Omega \setminus \overline{D}$ is 1. Then the electric potential function $u(x)$ in $\Omega \setminus \overline{D}$ satisfies the following boundary problem :

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{D} \\ \partial_\nu u = 0 & \text{on } \partial D \\ u = f & \text{on } \partial\Omega \end{cases}$$

where ν is the outer unit normal to ∂D.

This problem has a unique solution $u(f) \in H^1(\Omega \setminus \overline{D})$ for any $f \in H^{\frac{1}{2}}(\partial\Omega)$. The boundary measurements are given by the Dirichlet-to-Neumann map (DN-map), defined by

$$\Lambda_D f = \partial_\nu u(f)|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega) \quad (f \in H^{\frac{1}{2}}(\partial\Omega))$$

Inverse Problem. From given Λ_D, reconstruct the shape and the location of D.

We can reconstruct the convex hull of D by the enclosure method. In the method the special solution of Laplace equation in Ω, called complex geometrical optics (CGO) solution, plays the central role. It is defined by

$$v(x; \tau, t, \omega) = e^{\tau(x\cdot\omega - t)} e^{i\tau x\cdot\omega^\perp}$$

where $\tau (> 0)$ and $t \in \mathbb{R}$ are parameters, and $\omega, \omega^\perp \in S^1$ satisfy $\omega \cdot \omega^\perp = 0$.

Using this function, we define the indicator function :

Definition 1.1 (indicator function). Define

$$I_\omega(\tau, t) := \langle (\Lambda_D - \Lambda_\emptyset) v|_{\partial\Omega}, \overline{v}|_{\partial\Omega} \rangle$$

where $\Lambda_\emptyset$ is the DN-map for $D = \emptyset$

Note that the indicator function can be calculated from the boundary measurements.

Ikehata proved that we can reconstruct the convex hull of D by investigating the asymptotic behavior of the indicator function. Note that we can reconstruct the convex hull of D by reconstructing the function $h_D(\omega) := \sup_{x\in D} x \cdot \omega$ $(\omega \in S^1)$.

Theorem 1.2 (Ikehata). *$h_D(\omega)$ can be reconstructed. More precisely,*

$$t > h_D(\omega) \Longrightarrow \lim_{\tau\to\infty} I_\omega(\tau,t) = 0,$$

$$t = h_D(\omega) \Longrightarrow \lim_{\tau\to\infty} I_\omega(\tau,t) \neq 0.$$

2 Main result

We want to reconstruct the information about D better than its convex hull by using other CGO-solution of the Laplace equation. The CGO-solution with nonlinear phase was obtained for more general equation by Kenig, Sjöstrand, and Uhlmann [2]. In particular, the following function can be applied to our case.

Definition 2.1.

$$v(x;\tau,t,x_0,\omega) := e^{\tau(t-\log|x-x_0|)}e^{-i\tau f(x)} \quad (x\in\overline{\Omega})$$

where $\tau(>0)$ and $t\in\mathbb{R}$ are parameters, $x_0\in\mathbb{R}^2\setminus\overline{\Omega}$, $\omega\in S^1$, and $f(x)$ is a function given by

$$f(x) = d_{S^1}\left(\frac{x-x_0}{|x-x_0|},\omega\right)$$

where $d_{S^1}(\,\cdot\,,\,\cdot\,)$ is the metric function on S^1.

v satisfies the Laplace equation in Ω.

Lemma 2.2. *Assume that there exists $\omega_0\in S^1$ such that $\{x\in\mathbb{R}^2\ ;\ x-x_0 = k\omega_0,\ k>0\}\cap\partial\Omega=\emptyset$, then $v(x;\tau,t,x_0,\omega_0)$ satisfies $\triangle v = 0$ in Ω.*

Using this special solution, we define the indicator function :

Definition 2.3.

$$I_{x_0}(\tau,t) := \langle(\Lambda_D-\Lambda_\emptyset)v|_{\partial\Omega},\overline{v}|_{\partial\Omega}\rangle$$

We define a function $h_D(x_0)$ by $h_D(x_0) := \inf_{x\in D}\log|x-x_0|$ $(x_0\in\mathbb{R}^2\setminus\overline{\Omega})$.

Theorem 2.4. *Assume that there exists $\omega_0\in S^1$ such that $\{x\in\mathbb{R}^2\ ;\ x-x_0=k\omega_0,\ k>0\}\cap\partial\Omega=\emptyset$, then we can reconstruct $h_D(x_0)$. More precisely,*

$$t < h_D(x_0) \Longrightarrow \lim_{\tau\to\infty} I_\omega(\tau,t) = 0,$$

$$t = h_D(x_0) \Longrightarrow \lim_{\tau\to\infty} I_\omega(\tau,t) \neq 0.$$

Note that $e^{h_D(x_0)}$ is the distance between x_0 and D. Therefore, $D\subset\{x\in\Omega\ ;\ |x-x_0|\geq e^{h_D(x_0)}\}$. Setting $M=\{x_0\in\mathbb{R}^2\setminus\overline{\Omega}\ ;\ x_0$ satisfies assumption in theorem$\}$, we have

$$D\subset\bigcap_{x_0\in M}\{x\in\Omega\ ;\ |x-x_0|\geq e^{h_D(x_0)}\}\subset ch(D)$$

where $ch(D)$ is the convex hull of D.

References

[1] Ikehata,M., Enclosing a polygonal cavity in a two dimensional bounded domain from Cauchy data, Inverse Problems, 15(1999), 1231-1241

[2] C.E.Kenig, J.Sjöstrand, and G.Uhlmann, The Calderón problem with partial data, Ann. of Math.(to appear), arXiv:math.AP/0405486.