Existence and uniqueness for geometric solutions of implicit second order ordinary differential equations

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We are going to talk about conditions for existence and uniqueness for geometric solutions of implicit second order ordinary differential equations.

All manifolds and map germs considered here are differentiable of class C^∞, unless stated otherwise.

An implicit second order ordinary differential equation (or, briefly, an equation) is given by the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

where F is a smooth function of the independent variable x, of the function y. It is natural to consider $F = 0$ as being defined on a subset in the space of 2-jets of functions of one variable, $F : U \to \mathbb{R}$ where U is an open subset in $J^2(\mathbb{R}, \mathbb{R})$. Throughout this talk, we assume that 0 is a regular value of F.

If $F = 0$ satisfies $F_q = \partial F/\partial q \neq 0$ at z_0, by the implicit function theorem, we can locally rewrite this equation in the form $q = f(x, y, p)$, where f is a smooth function. This explicit form $q = f(x, y, p)$ is more convenient than the original one, because there exists the classical existence and uniqueness for (smooth) solutions.

Here we say that a **smooth solution** (or, a **classical solution**) of $F = 0$ around z_0 is a smooth function germ $y = f(x)$ at a point t_0 such that $(t_0, f(t_0), f'(t_0), f''(t_0)) = z_0$ and $F(x, f(x), f'(x), f''(x)) = 0$. In other words, there exists a smooth function germ $f : (\mathbb{R}, t_0) \to \mathbb{R}$ such that $j^2 f : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$. It is easy to check that the map $j^2 f$ satisfies $(j^2 f)^* \alpha_1 = (j^2 f)^* \alpha_2 = 0$ (i.e. $j^2 f$ is an Engel immersion germ).

More generally, a **geometric solution** of $F = 0$ around z_0 is an Engel immersion germ $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$, namely, $\gamma' \neq 0$, $\gamma^* \alpha_1 = \gamma^* \alpha_2 = 0$ and $F(\gamma(t)) = 0$.

By definition, the classical solution is also a geometric solution. Hence for explicit equations, there exists the unique geometric solution around z_0.

For implicit second order ordinary differential equations, however, existence and uniqueness for geometric solutions does not hold in general. In this talk, we give a sufficient condition about existence and uniqueness for geometric solutions of implicit second order ordinary differential equations.