Effective potentials of quantum particles interacting with quantum fields

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1. Introduction

Quantum field theory has predicted many physical phenomena and its validity has been verified experemintally, since it was born. Almost all of quantum field theoretical models, however, cannnot be solved exactly and one needs to use suitable approximation strategies. In physics, such an approximation strategy is often based on formal (non-rigorous) perturbation calculations. Many attempts have been made to give a mathematically rigorous basis for such calculations.

As one of the attempts, Arai [1] established an abstract theory of scaling limits on self-adjoint operators. Applying the theory to a Hamiltonian of the *Nelson model*, describing nonrelativistic quantum particles interacting with a scalar (Bose) field, Hiroshima [2] derived a Schrödinger Hamiltonian with an effective potential called the *Yukawa potential*. The potential was originally introduced by Yukawa to describe the nuclear force binding two (spinless) nucleons, which caused by the exchange of one (spinless) pion. Acutually, the nucleon-nucleon interaction is independent of whether the nucleons are nutrons or protons. Hence we have to treat both of a nutron and a proton as a nucleon and regard a nucleon as having a nutron state and a proton state. The degree of freedom of whether a nucleon is a nutron or a proton is called the *isospin* of the nucleon. Pions, also, have an isospin of whether pions are charged or not. Moreover, each nucleon has spin 1/2. The Yukawa potential, however, does not inclued any effect of the spin and the isospins, since such an effect was neglected in the Nelson model. In this talk, to treat models including such a degrre of freedom, we generalize the Nelson model, from which we derive a Schrödinger Hamiltonian with an effective potential.

2. Model and the Result

We now introduce a generalization of the Nelson model describing N particles coupled to a Bose field with some degrees of freedom like a spin or an isospin, whose Hamiltonian is given by

$$H = -\frac{1}{2M}\Delta \otimes I + I \otimes H_{\mathrm{f}} + gH_{\mathrm{I}}$$

and acting on the state space $L^2(\mathbb{R}^{dN}; \mathcal{H}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K})$, where $M > 0$ denotes the mass of the particles, Δ the generalized Laplacian on $L^2(\mathbb{R}^{dN})$. For simplicity, let $\mathcal{K} = L^2(\mathbb{R}^d)$. The free Hamiltonian of the bose field H_{f} is given by

$$H_{\mathrm{f}} = \int_{\mathbb{R}^d} \omega(k) a(k)^* a(k) dk,$$

which acts on the boson Fock space $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ over one boson Hilbet space $\mathcal{K}$, where ω is the disparsion relation of one boson and $a(k)$ the annihilation operator. The adjoint $a(k)^*$, called the creation operator, and $a(k)$ obey the canonical commutation relations:

$$[a(k), a(k')] = [a(k)^*, a(k')^*] = 0, \ [a(k), a(k')^*] = \delta(k - k').$$

The constant $g \in \mathbb{R}$ denotes a coupling constant which represents the strength of the interaction and H_{I} an interaction between the particles and the Bose field of the form

$$H_{\mathrm{I}} = \sum_{j=1}^{J} B_j \otimes \phi(\lambda_j(x)),$$

where $J \in \mathbb{N}$, B_j are bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ of some degrees of freedom, and

$$\phi(\lambda) = \frac{1}{\sqrt{2}} \int (\lambda^*(k)a(k) + \lambda(k)a(k)^*)dk, \ \ \lambda \in L^2(\mathbb{R}^d).$$

In the Hamiltonian of the Nelson model B_j are constants. In this talk, for simplicity, we set $\omega(k) = \sqrt{k^2 + m^2}$ $(m \geq 0)$ and let λ_j a $\mathcal{K}$-valued function of the form

$$\lambda_j(x) = e^{-ik\cdot x_j} v, \qquad \text{a.e.} x = (x_1, \cdots, x_N) \in \mathbb{R}^{dN}$$

where $v \in \mathcal{K}$ satisfies

$$\omega^{1/2} v, \ \ \omega^{-3/2} v \in \mathcal{K}$$

with

$$v(k)^* = v(-k), \ \ \text{a.e.} k \in \mathbb{R}^d.$$

A scaled Hamiltonian of H is introduced by

$$H(\Lambda) = -\frac{1}{2M} \Delta \otimes I + \Lambda^2 I \otimes H_{\mathrm{f}} + g\Lambda H_{\mathrm{I}}, \ \ \Lambda > 0.$$

We consider a limit of $H(\Lambda)$ as $\Lambda \to \infty$. We set

$$V_{\mathrm{eff}} = -\frac{g^2}{2} \sum_{j,k} B_j B_k \int \frac{|v(k)|^2}{\omega(k)} e^{ik\cdot(x_j - x_k)} dk, \qquad \text{a.e.} x \in \mathbb{R}^{dN}.$$

Theorem 1 *Let $t \in \mathbb{R}$. Then,*

$$\text{s-}\lim_{\Lambda\to\infty} e^{-itH(\Lambda)}(I \otimes P_0) = e^{-it(-\frac{1}{2M}\Delta + V_{\mathrm{eff}})} \otimes P_0, \tag{1}$$

where P_0 denotes the othogonal projection of $\ker H_{\mathrm{f}}$.

We call the limit (1) the scaling limit of $H(\Lambda)$. Physically, the unitary group $\{e^{-itH(\Lambda)}\}_{t\in\mathbb{R}}$ (resp. $\{e^{-it(-\frac{1}{2M}\Delta + V_{\mathrm{eff}})}\}_{t\in\mathbb{R}}$) describes the time-evolution of the system whose Hamiltonian is the generator $H(\Lambda)$ (resp. $-\frac{1}{2M}\Delta + V_{\mathrm{eff}}$). A vector belonging to the subspace $\ker H_{\mathrm{f}}$ represents the vacuum of the Bose field. Therefore one obtains a Schrödinger Hamiltonian with the effective potential V_{eff} in the vacuum of the Bose field in the sense of the limit for the time-evolution of the system.

References

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[2] F. Hiroshima, Weak coupling limit with a removal of an ultraviolet cutoff for a Hamiltonian of particles interacting with a massive scalar field, *Inf. Dimen. Anal. Quantum Prob. Relat. Top.* **1** (1998), 407-423.