Enhanced binding for N-particle system interacting with a scalar bose field [1]

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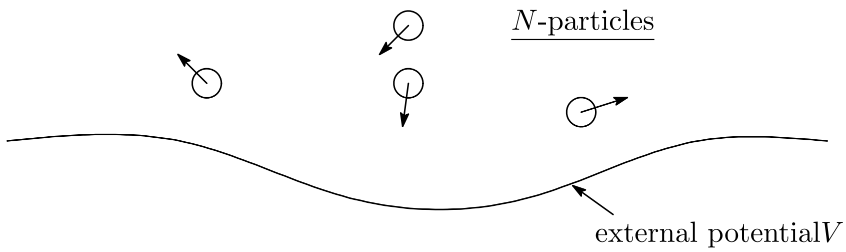
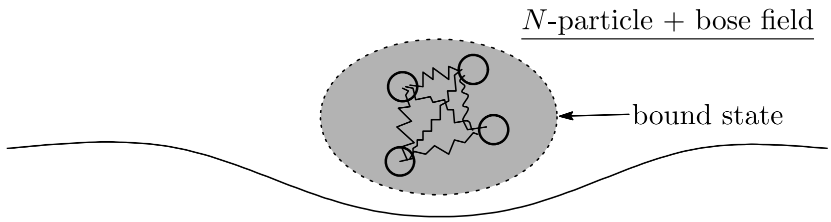
We consider a system of an N-particle system interacting with a scalar bose field for $N \geq 2$. For simplicity, we don't consider the statistics(bose and fermion statistics) for the N particles. The N particles moving on $\mathbb{R}^3$ under an influence of an external scalar potential $V : \mathbb{R}^3 \to \mathbb{R}$. The Hilbert space for the N particles is $L^2(\mathbb{R}^{3N})$ and Hamiltonian for the N particles is defined by

$$H_p := -\sum_{j=1}^{N} \frac{1}{2m} \triangle_j + \sum_{j=1}^{N} V(x_j), \tag{1}$$

where $m > 0$ denotes the mass of the particle, $(x_1, \dots, x_N) \in \mathbb{R}^{3N}$ is position of the N-particles and $\triangle_j$ is Laplace operator on x_j. In the Hamiltonian H_p, there is no internal force between the N particles. Namely, in the Hamiltonian H_p, the N particles move independently each other. The N particles should interact each other through a bose field. In this report, we assume that the external scalar potential V is relatively compact with respect to $-\triangle$ and sufficiently shallow such that H_p has no eigenvalue:



Next, we suppose that each particle interacts with a scalar bose field. These particles interact each other through the bose field, although any particle can't interacts the other particle directly. Under this situation, the N particles interact attractively and the N-particle system including the bose field should have a ground state:



The phenomena like this is called *the enhanced binding.*

In the following, we give the precise definition of our system. The particle Hamiltonian H_p is self-adjoint and bounded from below because V is relatively compact with respect to $-\triangle$.

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Hilbert space for the scalar bose field is given by

$$\mathcal{F}_{\mathrm{b}} := \bigoplus_{n=0}^{\infty} \left[\bigoplus_{\mathrm{sym}}^{n} L^2(\mathbb{R}^3) \right], \tag{2}$$

where $\otimes_{\mathrm{sym}}^{n}$ denotes the n-fold symmetric tensor product with $\otimes_{\mathrm{sym}}^{0} := \mathbb{C}$. Any boson state is described by a unit vector in $\mathcal{F}_{\mathrm{b}}$. For any $f \in L^2(\mathbb{R}^3)$, the creation operator $a(f)^*$ is defined by

$$\mathrm{Dom}(a(f)^*) := \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}} \Big| \sum_{n=1}^{\infty} n \| S_n f \otimes \Psi^{(n-1)} \|^2_{\otimes_{\mathrm{sym}}^{n} L^2(\mathbb{R}^3)} < \infty \right\}, \tag{3}$$

$$(a(f)^*\Psi)^{(n)} := \sqrt{n} S_n f \otimes \Psi^{(n-1)}, \quad n = 1, 2, \ldots, \tag{4}$$

$$(a(f)^*\Psi)^{(0)} := 0, \tag{5}$$

where S_n denotes the symmetrization operator. A special unit vector $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}_{\mathrm{b}}$ is called the vacuum vector. $a(f)^*$ is a closed operator acting on $\mathcal{F}_{\mathrm{b}}$ and its adjoint $a(f) := (a(f)^*)^*$ is called the annihilation operator. One can easily show that all annihilation operator vanish the vacuum: $a(f)\Omega = 0$. Obviously, $a(f)^*$ is linear in $f \in L^2(\mathbb{R}^3)$ and $a(f)$ is antilinear in f. We informally write as follows:

$$a(f)^* = \int_{\mathbb{R}^3} f(k) a(k)^* \mathrm{d}k, \quad a(f) = \int_{\mathbb{R}^3} f(k)^* a(k) \mathrm{d}k. \tag{6}$$

It is known that the integral kernels of $a(f)^*$ and $a(f)$ satisfy the canonical commutation relations:

$$[a(k), a(k')^*] = \delta(k - k'), \quad [a(k), a(k')] = [a(k)^*, a(k')^*)] = 0. \tag{7}$$

The Hamiltonian for the free boson is defined by

$$H_f := \int_{\mathbb{R}^3} |k| a(k)^* a(k) \mathrm{d}k. \tag{8}$$

H_f is a nonnegative self-adjoint operator on $\mathcal{F}_{\mathrm{b}}$ and the vacuum vector Ω is a unique eigenvector of H_f.

The Hilbert space for the total system is defined by

$$\mathcal{H} := L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_{\mathrm{b}}, \tag{9}$$

and the Hamiltonian for the N particles and the bose field is defined by

$$H(\kappa) := H_p \otimes I + \kappa^2 H_f \otimes I + \kappa H_I, \tag{10}$$

with

$$H_I := \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{N} \int_{\mathbb{R}^3} (a(k)^* \overline{\hat{\lambda}(k)} e^{-ikx_j} + a(k) \hat{\lambda}(k) e^{ikx_j}) \mathrm{d}k, \tag{11}$$

where $\alpha \in \mathbb{R}$ is a coupling constant, $\kappa > 0$ is a scaling parameter and $\hat{\lambda} \in L^2(\mathbb{R}^3)$ is a function. Typical example of $\hat{\lambda}$ is the function $|k|^{-1/2} \chi_{a,\Lambda}(k)$ where $\chi_{a,\Lambda}$ is the characteristic function of $\{k \in \mathbb{R}^3 | a < |k| < \Lambda\}$ and $0 \le a < \Lambda$ are positive constants. In the Hamiltonian $H(\kappa)$, the self-adjoint operator $H_p \otimes I + \kappa^2 H_f \otimes I$ is regarded as a unperturbed Hamiltonian, and κH_I describes the interaction between the N particles and the bose field. It is known that the self-adjoint operator H_I is infinitesimally small with respect to the unperturbed Hamiltonian, and hence $H(\kappa)$ is self-adjoint operator. Note that, since H_p has no eigenvalue, the unperterbed Hamiltonian has no eigenvalue. In the total system, there is no direct interaction between the N particles, but the N particles interact each other by the bose field. The effective potential between the N particles is given by:

$$V_{\text{eff}}(x_1, \ldots, x_N) := -\frac{\alpha^2}{2} \sum_{i \neq j}^{N} \operatorname{Re} \int_{\mathbb{R}^3} \frac{|\hat{\lambda}(k)|^2}{|k|} e^{-ik(x_i - x_j)} \mathrm{d}k. \tag{12}$$

So we define the effective Hamiltonian for the N particles as follows:

$$H_{\text{eff}} := H_p + V_{\text{eff}}. \tag{13}$$

For a self-adjoint operator T which is bounded from below, we say that T has a ground state if and only if $\inf \sigma(T)$ is an eigenvalue of T. In addition, when $\inf \sigma(T)$ is an discrete spectrum of T, we say that T has a discrete ground state.

We set two assumptions:

[V] There exists a constant $\alpha_c > 0$ such that H_{eff} has a discrete ground state.

[L] (i) $\hat{\lambda}$ is a real function and $|k|^{-1/2}\hat{\lambda} \in L^2(\mathbb{R}^3)$. (ii) There exist an openset $S \subset \mathbb{R}^3$ such that $\bar{S} = \operatorname{supp} \hat{\lambda}$ and $\hat{\lambda}$ is continuously differentiable in S. (iii) For all $R > 0$, the openset $\{k \in S | |k| < R\}$ satisfies a cone-property. (iv) For all $p \in [1, 2)$ and all $R > 0$, $|\nabla \hat{\lambda}| \in L^p(S_R)$.

Under this conditions, the enhanced binding occurs:

Theorem 1. Let $|k|^{-1}\hat{\lambda} \in L^2(\mathbb{R}^3)$. Assume [V], [L]. Fix a sufficiently large κ. Then there exist a constant α_c, for $\alpha_c < |\alpha| < \alpha_c(\kappa)$, $H(\kappa)$ has a ground state, where $\alpha_c(\kappa)$ is a constant but possibly infinity.

The scaling parameter κ in Theorem 1 can be regarded as a dummy and absorbed into m, V and $\hat{\lambda}$. We divide $H(\kappa)$ by κ^2:

$$\hat{H} := \sum_{j=1}^{N} \left(-\frac{1}{2\hat{m}} \triangle_j + \hat{V}(x_j) \right) \otimes I + I \otimes H_f + \hat{H}_I = \kappa^{-2} H(\kappa), \tag{14}$$

where $\hat{m} := m\kappa^2$, $\hat{V} := V/\kappa$ and $\hat{H}$ is defined by H_I with $\hat{\lambda}$ replaced by $\hat{\lambda}/\kappa$. Hence the condition that κ is sufficiently large implies that the mass of the particle is large, the external potential is shallow and the coupling function is small.

Remark 1. If there is no interaction between particles, the N particles is influenced only by the external scalar potential V. In this case, a shallow potential V can not trap these particles. But if these particles attractively interact through an effective potential derived from a scalar bose field, particles close up and behave just like as one particle with mass Nm. This *one* *particle* may feel the force $-N\nabla V$. If N is large enough, this one particle feels the potential NV strongly, and finally it will be trapped. Theorem 1 justifies this intuition.

Outline of proof of Theorem 1
We define a unitary transformation T as follows

$$T := \exp\Big(- i\frac{\alpha}{\kappa}\sum_{j=1}^{N} \pi_j\Big), \tag{15}$$

$$\pi_j := \frac{i}{\sqrt{2}}\int_{\mathbb{R}^3} |k|^{-1}(a^*(k)\overline{\hat{\lambda}(k)}e^{-ikx_j} + a(k)\hat{\lambda}(k)e^{ikx_j}). \tag{16}$$

As a first step of proof, we transform $H(\kappa)$ by T:

$$T^{-1}H(\kappa)T = H_{\text{eff}}\otimes I + \kappa^2 I\otimes H_f + H'(\kappa), \tag{17}$$

where

$$H'(\kappa) := \sum_{j=1}^{N}\Big\{\frac{\alpha}{2m\kappa}((-i\nabla_j\otimes I)\widetilde{\phi}_j + \widetilde{\phi}_j(-i\nabla\otimes I)) + \frac{\alpha^2}{2m\kappa^2}\widetilde{\phi}_j^2\Big\} - \frac{\alpha^2}{2}N\||k|^{-1/2}\hat{\lambda}\|^2, \tag{18}$$

$$\widetilde{\phi}_j := \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\frac{k}{|k|}\Big(a^*(k)\overline{\hat{\lambda}(k)}e^{-ikx} + a(k)\hat{\lambda}(k)e^{ikx}\Big)\mathrm{d}k. \tag{19}$$

This transformation is a key of the proof. Note that the last term in $H'(\kappa)$ is a constant and the other term in $H'(\kappa)$ is small if κ is sufficiently large. Next, we set $C_N := \{1,\dots,N\}$ and for $\beta\subset C_N$, we define

$$H^0(\beta) := \frac{1}{2m}\sum_{j\in\beta}\Big(- i\nabla_j\otimes I - \frac{\alpha}{\kappa}\widetilde{\phi}_j\Big)^2 + \kappa^2 I\otimes H_f + V_{\text{eff}}(\beta)\otimes I, \tag{20}$$

$$V_{\text{eff}}(\beta) := \begin{cases} -\dfrac{\alpha^2}{4}\displaystyle\sum_{i,j\in\beta.\, i\neq j}\int_{\mathbb{R}^3}|k|^{-1}|\hat{\lambda}(k)|^2 e^{-ik(x_i-x_j)}\mathrm{d}k, & \text{for } |\beta|\geq 2. \\ 0 & \text{for } |\beta| = 0,1. \end{cases} \tag{21}$$

$$H^V(\beta) := H^0(\beta) + \sum_{j\in\beta}V_j\otimes I. \tag{22}$$

$H^0(\beta)$ and $H^V(\beta)$ are self-adjoint operators acting on $L^2(\mathbb{R}^{3|\beta|})\otimes\mathcal{F}_{\text{b}}$. $H^0(\beta)$ and $H^V(\beta)$ are cluster Hamiltonian for a cluster β. We set

$$E^V(\kappa) := \inf\sigma(H^V(C_N)), \quad E^V(\kappa,\beta) := \inf\sigma(H^V(\beta)), \tag{23}$$

$$E^0(\kappa,\beta) := \inf\sigma(H^0(\beta)), \quad E^V(\kappa,\emptyset) := 0. \tag{24}$$

The lowest two cluster threshold $\Sigma^V(\kappa)$ is defined by

$$\Sigma^V(\kappa) := \min\{E^V(\kappa,\beta) + E^0(\kappa,\beta^c) | \beta\subsetneqq C_N\}. \tag{25}$$

To extablish the existence of ground state of $H(\kappa)$, we use the next proposition:
Proposition([0]). Let $\Sigma^V(\kappa) - E^V(\kappa) > 0$. Then $H(\kappa)$ has a ground state.
In order to use this proposition, we need the condition [L]. Finally, we can prove Theorem 1 by estimating the lowest two cluster threshold $\Sigma^V(\kappa)$ and the lowest energy $E^V(\kappa)$.

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In the following, we explain the examples of V and V_{eff} in the condition [V]. Let

$$h^V(\alpha) := \sum_{j=1}^{N} \left(-\frac{1}{2m} \triangle_j + V(x_j) \right) + \alpha^2 \sum_{j \neq l}^{N} W(x_j - x_l), \tag{26}$$

which acts on $L^2(\mathbb{R}^{3N})$. We assume (W1)-(W3) below:

[W1] V is relatively compact with respect to the 3-dimensional Laplacian $\triangle$ and $-\triangle/2m + V$ has no eigenvalue.

[W2] W satisfies that $-\infty < W(0) < \operatorname*{ess.inf}_{|x|<\epsilon} W(x) < \operatorname*{ess.inf}_{|x|>\epsilon} W(x)$ for all $\epsilon > 0$.

[W3] $-\triangle/(2Nm) + NV$ has a discrete ground state:

$$\inf \sigma \left(-\frac{1}{2Nm} \triangle + NV \right) \in \sigma_{\text{disc}} \left(-\frac{1}{2Nm} \triangle + NV \right). \tag{27}$$

The following theorem ensures that the potentials which satisfy the condition [V] exist:

Theorem 2. Assume [W1], [W2] and [W3]. Then there exists $\alpha_c > 0$ such that for all α with $|\alpha| > \alpha_c$, $h^V(\alpha)$ has a discrete ground state.

Remark 2. The condition [W1] means that the external potential V is sufficiently shallow and can not trap the particle. The condition [W2] means that the internal potential $W(x)$ has the lowest value at the origin $x = 0$ and $W(0)$ is finite. Namely, $W(x_j - x_l)$ is an attractive potential. [W3] implies that the one body Schrödinger operator with mass Nm and potential Nm has a discrete ground state.

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