

Orthogonal polynomials on the unit circle associated with a rigid function

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Let $\partial\mathbb{D}$ be the unit circle $\{z; |z| = 1\}$ in $\mathbb{C}$, and $\mathbb{D}$ the open unit disc $\{z; |z| < 1\}$. Let μ be a probability measure on $\partial\mathbb{D}$ which is not supported on a finite number of points. Then, by applying the Gram–Schmidt method to $1, z, z^2, \dots, z = e^{i\theta}$, we can define monic *orthogonal polynomials on the unit circle*:

$$\Phi_n(z) = z^n + \text{lower order}, \quad n = 0, 1, \dots, \quad (\Phi_m, \Phi_n)_\mu = 0, \quad m \neq n.$$

These obey the Szegő recursion formula

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n z^n \overline{\Phi_n(1/\bar{z})}, \quad n = 0, 1, \dots,$$

with $\alpha_n = -\overline{\Phi_{n+1}(0)} \in \mathbb{D}$. The parameters α_n are called the *Verblusky coefficients* of $d\mu$. See [13] for background.

A function $g \in H^1$ is said to be *rigid* (or *strongly outer*) if it is determined by its argument up to a positive constant factor: If $k \in H^1$ satisfies $k/|k| = g/|g|$ a.e. on $\partial\mathbb{D}$, then $k = ag$ for some $a > 0$, see, e.g., [8, 9, 10, 11]. A rigid function g is outer, and hence it may be expressed as $g = h^2$ with an outer function $h \in H^2$. For $\phi \in L^\infty$, we define the Toeplitz operator T_ϕ on H^2 by $T_\phi f = P(\phi f)$, where P is the orthogonal projection operator of L^2 onto H^2 . The square of $h \in H^2$ is rigid if and only if $h \in$ the closure of $T_{h/\bar{h}}H^2$ in L^2 , cf.[2].

Theorem 1 ([1, 7, 9]). *We have*

$$h\Phi_n \in \text{the closure of } [z^n(h/\bar{h})H^2 + \overline{zH^2}] \text{ in } L^2, \quad n = 0, 1, \dots,$$

if and only if $d\mu = \frac{1}{2\pi}|h|^2 d\theta$ *and* h^2 *is rigid.*

Theorem 2. *We have*

$$h\Phi_n \in z^n(h/\bar{h})H^2 + \overline{zH^2}, \quad n = 0, 1, \dots,$$

if and only if $d\mu = \frac{1}{2\pi}|h|^2 d\theta$ *and* $h \in T_{h/\bar{h}}H^2$.

Proposition 3. *Let* $h \in H^2$. *We have* $h \in T_{h/\bar{h}}H^2$ *and* $\|h\|_2 \leq 1$ *if and only if*

$$h = \frac{1}{\sigma - z\tau}$$

for some $\sigma, \tau \in H^2$ *such that* σ *is outer and* $|\sigma|^2 - |\tau|^2 = 1$ *a.e. on* $\partial\mathbb{D}$.

For simplicity, we assume that $h(0) > 0$. For $\phi \in L^\infty$, we define the Hankel operator H_ϕ from H^2 to $\overline{zH^2}$ by $H_\phi f = Q(\phi f)$, where Q is the orthogonal projection operator of L^2 onto $\overline{zH^2}$. Let $n = 0, 1, \dots$ and write $H_n = H_{z^n(h/\bar{h})}$. Putting

$$\varphi_n^{2k} = \{H_n^* H_n\}^k 1, \quad \varphi_n^{2k+1} = \{H_n^* H_n\}^k H_n^* \bar{z},$$

we define $\sigma_n^m, \tau_n^m \in H^2$ by

$$\sigma_n^m = h(0) \sum_{k=0}^m \varphi_n^{2k}, \quad \tau_n^m = -h(0) \sum_{k=0}^m \varphi_n^{2k+1}.$$

In the case of Theorem 1, it follows that

$$\Phi_n = \lim_{m \rightarrow \infty} [\bar{h}^{-1} z^n \sigma_n^m + h^{-1} \overline{z \tau_n^m}] \text{ in } L^2(\partial \mathbb{D}, d\mu).$$

In the case of Theorem 2, there are $\sigma_n = \lim_{m \rightarrow \infty} \sigma_n^m$ and $\tau_n = \lim_{m \rightarrow \infty} \tau_n^m$ in H^2 , and we have

$$\Phi_n = \bar{h}^{-1} z^n \sigma_n + h^{-1} \overline{z \tau_n},$$

as well as

$$(I - H_n^* H_n) \sigma_n = h(0) \quad (I - H_n^* H_n) \tau_n = -h(0) H_n^* \bar{z}$$

(cf. [12]). Also, $\sigma = \sigma_0$ and $\tau = \tau_0$ hold for $\sigma, \tau \in H^2$ in Proposition 3.

The coefficients $\varphi_{n,j}^k$ in $\varphi_n^k(z) = \sum_{j=0}^\infty \varphi_{n,j}^k z^j$ can be expressed in terms of

$$\varphi_j = \frac{1}{2\pi} \int_{\partial D} e^{-ij\theta} (\bar{h}/h) d\theta,$$

in view of the recursion $\varphi_n^0 = 1$, $\varphi_n^{k+1} = H_n^* \overline{z \varphi_n^k}$, whence

$$\varphi_{n,j}^0 = \delta_{0j}, \quad \varphi_{n,j}^{k+1} = \sum_{l=0}^\infty \varphi_{n+j+l+1} \overline{\varphi_{n,l}^k}, \quad k = 0, 1, \dots$$

The following representations of α_n and $\|\Phi_n\|_\mu$ are useful for studying their asymptotic behavior as $n \rightarrow \infty$, see [3]–[7].

Theorem 4. *If $d\mu = \frac{1}{2\pi} |h|^2 d\theta$ and h^2 is rigid, then*

$$\alpha_n = - \sum_{k=0}^\infty \left(\sum_{l=0}^\infty \varphi_{n+l+1} \overline{\varphi_{n+1,l}^{2k}} \right) = - \frac{\sum_{k=0}^\infty \varphi_{n,0}^{2k+1}}{\sum_{k=0}^\infty \varphi_{n,0}^{2k}},$$

$$\|\Phi_n\|_\mu^2 = |h(0)|^2 \sum_{k=0}^\infty \varphi_{n,0}^{2k}, \quad n = 0, 1, \dots$$

REFERENCES

- [1] P. Bloomfield, N. P. Jewell, and E. Hayashi, Characterizations of completely nondeterministic stochastic processes, *Pacific J. Math.* **107** (1983), no. 2, 307–317.
- [2] E. Hayashi, The solution sets of extremal problems in H^1 , *Proc. Amer. Math. Soc.* **93** (1985), no. 4, 690–696.
- [3] A. Inoue, Asymptotics for the partial autocorrelation function of a stationary process, *J. Anal. Math.* **81** (2000), 65–109.
- [4] A. Inoue, Asymptotic behavior for partial autocorrelation functions of fractional ARIMA processes, *Ann. Appl. Probab.* **12** (2002), no. 4, 1471–1491.
- [5] A. Inoue, What does the partial autocorrelation function look like for large lags?, preprint.
- [6] A. Inoue and Y. Kasahara, Partial autocorrelation functions of the fractional ARIMA processes with negative degree of differencing, *J. Multivariate Anal.* **89** (2004), no. 1, 135–147.
- [7] A. Inoue and Y. Kasahara, Explicit representation of finite predictor coefficients and its applications, *Ann. Statist.* **34** (2006), no. 2, 973–993.
- [8] T. Nakazi, Exposed points and extremal problems in H^1 , *J. Funct. Anal.* **53** (1983), no. 3, 224–230.
- [9] T. Nakazi, Kernels of Toeplitz operators, *J. Math. Soc. Japan* **38** (1986), no. 4, 607–616.
- [10] D. Sarason, Exposed points in H^1 . I. The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), 485–496, *Oper. Theory Adv. Appl.*, **41**, Birkhäuser, Basel, 1989.
- [11] D. Sarason, Kernels of Toeplitz operators. Toeplitz operators and related topics (Santa Cruz, CA, 1992), 153–164, *Oper. Theory Adv. Appl.*, **71**, Birkhäuser, Basel, 1994.
- [12] A. Seghier, Prédiction d’un processus stationnaire du second ordre de covariance connue sur un intervalle fini, *Illinois J. Math.* **22** (1978), no. 3, 389–401.
- [13] B. Simon, Orthogonal polynomials on the unit circle. Part 1. Classical theory. American Mathematical Society, Providence, RI, 2005.