BOUNDARY BEHAVIOR OF SUPERHARMONIC FUNCTIONS

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1. Definitions

Let Ω be a bounded (smooth) domain in $\mathbb{R}^n$, $n \geq 2$, and let $\delta_\Omega(x)$ denote the distance from x to the boundary $\partial\Omega$ of Ω. By $B(x,r)$ and $S(x,r)$ we denote the open ball and the sphere of center x and radius r, respectively. A function $h \in C^2(\Omega)$ is called *harmonic* on Ω if $\Delta h = 0$ in Ω, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. It is well known that the harmonicity is characterized in terms of the continuity and the spherical mean value equality. The superharmonicity is defined as follows. A function $u : \Omega \to (-\infty, +\infty]$ is called *superharmonic* on Ω if $u \not\equiv +\infty$, u is lower semicontinuous on Ω and u satisfies the spherical mean value inequality: for each $x \in \Omega$ and $0 < r < \delta_\Omega(x)$,

$$u(x) \geq \frac{1}{\sigma(S(x,r))} \int_{S(x,r)} u d\sigma,$$

where σ is the surface area measure on $S(x,r)$. Note that if $u \in C^2(\Omega)$ then u is superharmonic on Ω if and only if $-\Delta u \geq 0$ in Ω. It is well known that if u is superharmonic on Ω, then there exists a unique (Radon) measure μ_u on Ω such that

$$\int_\Omega \phi d\mu_u = -\int_\Omega u \Delta\phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

The measure μ_u is called the Riesz measure associated with u. If μ_u is absolutely continuous with respect to Lebesgue measure, then the density function is written as $-\Delta u$. For each $y \in \Omega$, a unique solution of

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega \quad \text{(distributions)}, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

is called the *Green function* for Ω with pole at y. Let μ be a measure on Ω. The function $\int_\Omega G(\cdot, y) d\mu(y)$ is called the *Green potential* of μ if it is superharmonic on Ω. It is well known as the Riesz decomposition theorem that every nonnegative superharmonic function u on Ω can be represented as

$$u(x) = h(x) + \int_\Omega G(x,y) d\mu(y) \quad \text{for } x \in \Omega,$$

where h is the greatest harmonic minorant of u on Ω and μ is the Riesz measure on Ω associated with u.

In the talk, we shall state several classical results concerning the boundary behavior of harmonic functions and Green potentials.

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2. Results

Let $p > 0$ be a constant and let us consider positive superharmonic functions u on Ω satisfying

$$0 \le -\Delta u \le u^p \quad \text{in } \Omega \tag{1}$$

(assuming that its Riesz measure is absolutely continuous with respect to Lebesgue measure). In what follows, we suppose that $n \ge 3$ and that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$. We can obtain the following boundary growth estimate.

Theorem 1. *If $0 < p \le (n+1)/(n-1)$, then every positive superharmonic function on Ω satisfying (1) enjoys*

$$u(x) \le A\delta_\Omega(x)^{1-n} \quad \textit{for } x \in \Omega,$$

where A is a constant depending only on u, p, Ω.

Corollary 1. *Assumptions are same as in Theorem 1. Then $u \in C^1(\Omega)$. Moreover, there exists a constant A depending only on u, p, Ω such that*

$$\sup_{B(x,r)} u \le A \inf_{B(x,r)} u$$

whenever $B(x, 8r) \subset \Omega$.

Corollary 2. *Assumptions are same as in Theorem 1. If, in addition, the greatest harmonic minorant of u on Ω is the zero function, then u has nontangential limit zero almost everywhere on $\partial\Omega$.*

The following shows the sharpness of $p \le (n+1)/(n-1)$ in Theorem 1.

Theorem 2. *Suppose that $p > (n+1)/(n-1)$. Let $\xi \in \partial\Omega$ and let β satisfy*

$$n - 1 < \beta < \begin{cases} \dfrac{2}{n-(n-2)p} & \textit{if } p < \dfrac{n}{n-2}, \\ \infty & \textit{if } p \ge \dfrac{n}{n-2}. \end{cases}$$

Then there exists a positive superharmonic function $u \in C^2(\Omega)$ satisfying (1) such that

$$\limsup_{x \to \xi} \delta_\Omega(x)^\beta u(x) > 0 \quad \textit{nontangentially.}$$

The following shows the sharpness of the growth rate $1 - n$ in Theorem 1.

Theorem 3. *Let $\xi \in \partial\Omega$. The following statements hold.*

(i) *If $0 < p < (n+1)/(n-1)$* [1]*, then the Lane-Emden equation*

$$-\Delta u = u^p \quad \textit{in } \Omega \tag{2}$$

has positive solutions $u \in C^2(\Omega)$ satisfying

$$u(x) \approx \frac{\delta_\Omega(x)}{|x-\xi|^n} \quad \textit{for } x \in \Omega. \tag{3}$$

(ii) *If $p \ge (n+1)/(n-1)$, then there are no positive solutions of (2) satisfying (3).*

[1] in addition, the case $p = 1$ requires the smallness of the size of Ω.