Research Summary

Max Wakefield

My research interests are focussed on the algebraic, geometric, and combinatorial properties of hyperplane arrangements. A hyperplane is a linear space whose dimension is one less than the dimension of the ambient vector space V. A hyperplane arrangement $\mathcal{A}$ is just a finite collection of hyperplanes. We can also define a multiarrangement $(\mathcal{A}, m)$ by assigning positive integer weights to each hyperplane with a multiplicity function $m : \mathcal{A} \to \mathbb{Z}_{>0}$. When $m \equiv 1$ then $(\mathcal{A}, m)$ is just a hyperplane arrangement and we will only write $\mathcal{A}$. Most of my research is focussed around the module of derivations $D(\mathcal{A}, m)$ of a multiarrangement, but I have also explored the existence of nets in $\mathbb{CP}^2$ and apolar algebras of hyperplane arrangements.

Let S be the symmetric algebra of the vector space V. Then the module of derivations (also called the module of logarithmic vector fields when $m \equiv 1$) is defined by $D(\mathcal{A}, m) = \{\theta \in \mathrm{Der}(S, S) | \theta(\alpha_H) \in S\alpha_M^{m(H)}$ for all $H \in \mathcal{A}\}$ where α_H is a linear form defining the hyperplane H. So, $D(\mathcal{A}, m)$ is a submodule of the free module $\mathrm{Der}(S, S) \cong S^\ell$ where ℓ is the dimension of V. However, $D(\mathcal{A}, m)$ is rarely free because, for example, it is known that for all generic arrangements (so $m \equiv 1$) $D(\mathcal{A}, m)$ is not free. We will say an arrangement is free if $D(\mathcal{A}, m)$ is a free S-module. If an arrangement is free, then we can define the exponents of $(\mathcal{A}, m)$, and write $\exp(\mathcal{A}, m)$, as the polynomial degrees of the elements of a homogeneous basis for $D(\mathcal{A}, m)$. The next theorem is the one of the most significant theorems in the theory of free arrangements.

Theorem 1 (Terao) *If $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (e_1, \ldots, e_\ell)$, then $\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(1 + e_i t)$ where $\pi(\mathcal{A}, t)$ is the Poincaré polynomial of the arrangement.*

Then Terao made the following conjecture.

Conjecture 2 (Terao) *For a fixed field, the freeness of $\mathcal{A}$ depends only on the intersection lattice.*

This conjecture is the central theme of my research. If the dimension of V is two, then $(\mathcal{A}, m)$ is always free. The main focus of my dissertation was to study the exponents of these multiarrangements in dimension two. The following is the main theorem of my dissertation.

Theorem 3 *There exists a Zariski open set in the moduli space of n weighted (with integer function m) points in $\mathbb{CP}^1$ such that if $\mathcal{A}$ is in this open set then $\exp(\mathcal{A}, m) = \left(\lfloor \frac{1}{2} \sum m(H) \rfloor, \lceil \frac{1}{2} \sum m(H) \rceil\right)$. Further, if the multiplicity function satisfies $m(H) < \sum_{H \neq H' \in \mathcal{A}} m(H')$ for all H and $\sum_{H \in \mathcal{A}} m(H) > 2n - 1$, then the Zariski open set is non-empty.*

In a paper with S. Yuzvinsky we prove Theorem 3 and we also proved many cases where the exponents do not depend on the position of the points. Then we used a main theorem of Yoshinaga that states that an arrangement in dimension three is free if and only if it's Poincaré polynomial factors and there is a restriction multiarrangement such that it's exponents are the coefficients of the Poincaré polynomial. Thus, if there is a multiplicity function for points in $\mathbb{CP}^1$ such that the exponents do not depend on the position of the points then any arrangement in dimension three that has a restriction with these same multiplicities then this arrangement will satisfy Terao's conjecture (i.e. all arrangements with isomorphic intersection lattice to this arrangement are all free or not free). We combined these results and the property of supersolvable lattices to prove the following theorem.

Theorem 4 *If $\mathcal{A}$ is an arrangement in dimension three such that there is a hyperplane H where the restriction to H gives a multiarrangement in dimension two where there is one multiplicity $m(H \cap H')$ such that $m(H \cap H') > \frac{1}{2}(|\mathcal{A}| - 3)$ then Terao's conjecture holds for $\mathcal{A}$.*

The following corollary is an immediate consequence of Theorem 4.

Corollary 5 *Terao's conjecture holds for arrangements of hyperplanes in dimension three of size less than eleven.*

Now, we briefly discuss apolar algebras of hyperplane arrangements. Let $S = \mathbb{C}[x_1, \ldots, x_\ell]$ be the polynomial ring in the variables $x_1, \ldots, x_\ell$ and $\bar{S} = \mathbb{C}[\partial_1, \ldots, \partial_\ell]$ be a polynomial ring in the variables $\partial_1, \ldots, \partial_\ell$. Then S is a $\bar{S}$-module by the action of differentiation (i.e. $\partial_i x_j = \delta_{ij}$). For $f \in S$ the apolar algebra of f is $\bar{S}/I(f)$ where $I(f) = \mathrm{Ann}_{\bar{S}}(f)$. $\bar{S}/I(f)$ is a zero-dimensional Gorenstein ring and we say $\mathcal{A}$ is a complete intersection when $\bar{S}/I(f)$ is a complete intersection algebra. It is known that $\bar{S}/I(f)$ where f is the defining polynomial of a reflection arrangement is a complete intersection and that every two dimensional arrangement is a complete intersection. So, it seems that maybe there is a connection between complete intersection arrangements and free arrangements. It was known that free arrangements were not necessarily complete intersections, but the converse was unknown. However, in my dissertation I exhibited arrangements that are complete intersections and not free.

In the academic year of 2006 I have studied the freeness of higher dimensional multiarrangements, even though they are more elusive. For example, in joint work with T. Abe and H. Terao we define the Poincaré polynomial for any multiarrangement and denote it by $\pi((\mathcal{A}, m), t)$. Unlike the case for arrangements the Poincaré polynomial of a multiarrangement is not necessarily invariant of the intersection lattice. However, we are still able to prove the generalization of Theorem 1 to multiarrangements.

Theorem 6 *If $(\mathcal{A}, m)$ is a free multiarrangement in dimension ℓ with $\exp(\mathcal{A}, m) = (e_1, \ldots, e_\ell)$, then $\pi((\mathcal{A}, m), t) = \prod_{i=1}^{\ell}(1 + e_i t)$.*

The main result of the paper containing Theorem 6 with T. Abe and H. Terao shows that there is a local to global relationship of the coefficients of the Poincaré polynomial of a multiarrangement. To state this theorem we need a little more notation. Let $L = L(\mathcal{A})$ be the intersection lattice of $\mathcal{A}$ where the elements are intersections of hyperplanes with the order as reverse inclusion and the rank function defined by codimension: $r(X) = \mathrm{codim}_V(X)$. Let $L_k = \{X \in L \mid r(X) = k\}$ be the elements in L of rank k. For any $X \in L$ let $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ and $m_X = m|_{\mathcal{A}_X}$. Define $C_p(X) \in \mathbb{Z}$ by $\pi((\mathcal{A}_X, m_X), t) = \sum_p^{\ell} C_p(X) t^p$. Now, we can state the main theorem.

Theorem 7 *For all $X \in L$ and p such that $0 \leq p \leq r(X)$, $C_p(X) = \sum_{Y \in L(\mathcal{A}_X)_p} C_p(Y)$.*

Theorems 6 and 7 provide a useful and quick method to check freeness of multiarrangements. Another effective method which can also be used to prove freeness of an arrangement is the addition deletion theorem for multiarrangements that is again joint work with T. Abe and H. Terao. Let $(\mathcal{A}, m)$ be a multiarrangement in dimension $\ell \geq 2$ and fix a hyperplane $H_0 \in \mathcal{A}$. Then the deletion of $(\mathcal{A}, m)$ by H_0 is $(\mathcal{A}', m')$ which is the same as $(\mathcal{A}, m)$ except the multiplicity of H_0 is decreased one. The restriction $(\mathcal{A}'', m'')$ allows for nearly any definition of m''. For example, M. Yoshinaga's Theorem described above uses the sum of all the hyperplanes through each intersection on H_0 as m''. However, this multiplicity does not yield itself well to addition deletion type theorems. Thus, with T. Abe and H. Terao, we created the following definition.

Definition 8 *For $X \in \mathcal{A}''$ the* e-multiplicity *$m^*(X)$ of X is the exponent (i.e. degree of generator) of the multiarrangement $(\mathcal{A}_X, m_X)$ corresponding to the derivation such that under any change of basis this derivation is never divisible by the defining form α_{H_0}.*

When $m \equiv 1$ then $m^*(X) = 1$ and hence this multiplicity generalizes the multiplicity used for the addition deletion theorem for arrangements. Now, we can state the addition deletion theorem for multiarrangements which is joint work with T. Abe and H. Terao.

Theorem 9 *Let $(\mathcal{A}, m)$ be a nonempty multiarrangement in an ℓ-dimensional vector space V, $H_0 \in \mathcal{A}$ and let $(\mathcal{A}, m), (\mathcal{A}', m')$, $(\mathcal{A}'', m^*)$ be the triple with respect to H_0. Then any two of the following statements imply the third:*

(i) $(\mathcal{A}, m)$ *is free with* $\exp(\mathcal{A}, m) = (e_1, \ldots, e_\ell)$.

(ii) $(\mathcal{A}', m')$ *is free with* $\exp(\mathcal{A}', m') = (e_1, \ldots, e_\ell - 1)$.

(iii) $(\mathcal{A}'', m^*)$ *is free with* $\exp(\mathcal{A}'', m^*) = (e_1, \ldots, e_{\ell-1})$.

Theorem 9 can be used to prove large classes of multiarrangements are free. For example, if the underlying arrangement $\mathcal{A}$ is supersolvable and the multiplicities m satisfy some inequalities then the multiarrangement $(\mathcal{A}, m)$ is free. However, since the e-multiplicity is very delicate the freeness of multiarrangements for even simple cases can be very difficult.

Publications
Max Wakefield

[1] (with S. Yuzvinsky) Derivations of an effective divisor on the complex projective line, to appear, *Transactions of the American Mathematical Society*.

[2] (with T. Abe and H. Terao) The characteristic polynomial of multiarrangements, arxiv math.AC/0611742.

[3] (with T. Abe and H. Terao) The e-multiplicity and addition-deletion theorems for multiarrangements, arxiv math.CV/0612739.

[4] (with C. Dunn, M. Miller, and S. Zwicknagl) Equivalence classes of Latin squares and nets in $\mathbb{CP}^2$, arxive math.CO/0703142.

Conference and Seminar Presentations

Max Wakefield

[1] **Mathematical Sciences Research Institute**, Arrangements of Hyperplanes program, "Derivations of Multiarrangements", September 2004.

[2] **Mathematical Sciences Research Institute**, Arrangements of Hyperplanes program, "Exponents of generic multi-2-arrangements", October 2004.

[3] **Mathematical Sciences Research Institute**, Arrangements of Hyperplanes program, "Derivation module of points with multiplicity on the projective line", November 2004.

[4] **Centro Stefano Franscini**, Ascona, Switzerland, Arrangements of Hyperplanes - Algebra, Combinatorics, Geometry and Topology, "Complete Intersection Apolar Algebras", May 2005.

[5] **A.M.S. Spring Eastern Sectional**, Durham, NH, Special Session on Arrangements and Configuration spaces, "On nets and Latin squares", April 2006.

[6] **Mathematical Sciences Research Institute**, Recent Developments in Arrangements and Configuration Spaces, "Degeneration Varieties", August 2006.

[7] **Hokkaido University**, Arrangements of Hyperplanes Seminar, Sapporo, Japan, "Derivations of an effective divisor on the complex projective line I", October 2006.

[8] **Hokkaido University**, Arrangements of Hyperplanes Seminar, Sapporo, Japan, "Derivations of an effective divisor on the complex projective line II", October 2006.

[9] **Hokkaido University**, Arrangements of Hyperplanes Seminar, Sapporo, Japan, "Derivations of an effective divisor on the complex projective line III", November 2006.

[10] **Hokkaido University**, Arrangements of Hyperplanes Seminar, Sapporo, Japan, "Derivations of an effective divisor on the complex projective line IV", November 2006.

[11] **Hokkaido University**, Mini-workshop on Arrangements of Hyperplanes, Sapporo, Japan, "The characteristic polynomial of a multiarrangement", December 2006.

[12] **A.M.S. National**, New Orleans, LA., Special Session on Arrangements and Related Topics, "The characteristic polynomial of a multiarrangement", January 2007.