# Research Report

of

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In my research I am mainly interested in PDEs of parabolic type, in particular in the *Navier-Stokes equations*. One way to examine this problem in the setting of Lebesgue spaces $L^q$ is to use the *semigroup method*. This approach starts with the investigation of the linearized Stokes equations and the associated Stokes operator $A_\Omega$. Here the notions of a holomorphic semigroup and maximal regularity play a fundamental role.

In my work together with André Noll (see [NS03]) we were able to prove that the Stokes operator belongs to $\mathcal{H}^\infty(L^q_\sigma(\Omega))$, the class of all operators that admit a bounded $H^\infty$-calculus on the space $L^q_\sigma(\Omega)$ for a large class of domains $\Omega \subseteq \mathbb{R}^n$. Among other useful properties this class implies in particular the important maximal regularity.

The content of our article also represents one part of my Ph.D Thesis [Saa03] which I finished in July 2003.

Moreover, in my thesis I proved resolvent estimates for the Stokes operator with Robin boundary conditions in $L^q(\mathbb{R}^n_+)$ not only for $1 < q < \infty$, but also in $L^\infty(\mathbb{R}^n_+)$, which lead to the fact that the Stokes operator generates a bounded holomorphic semigroup on the spaces $L^\infty_\sigma(\mathbb{R}^n_+)$, $C_{0,\sigma}(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. This seems to be rather surprising, since I also proved that a similar result for the Stokes operator with Robin boundary conditions in $L^1(\mathbb{R}^n_+)$ is not valid, except for the case of Neumann boundary conditions. Up to now there was no literature available, which tackles this problem in $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$ for this type of mixed boundary conditions. I am only aware of the article [DHP01], where the problem merely is considered for the case of Dirichlet boundary conditions. If $1 < q < \infty$ then it is even proved that the Stokes operator with Robin boundary conditions admits a bounded $H^\infty$-calculus on $L^q_\sigma(\mathbb{R}^n_+)$. The just mentioned results are also the content of the forthcoming articles [Saa04], [Saab].

Since my arrival here in Sapporo on October 1, 2003, I was working on the Stokes equations on noncylindrical space-time domains $Q := \bigcup_{t\in[0,\infty)} \Omega(t) \times \{t\}$. In this case also the domain $\Omega = \Omega(t)$ depends on time $t \in (0,\infty)$. For this problem I could also prove maximal regularity on $L^p(0,T;L^q(\Omega(t)))$ under suitable assumptions on the evolution of $\Omega(t)$. The regularity results in this direction seem to be rather new, since in the present literature only existence results in the Hilbert space case $L^2(Q)$ are available. The work also includes an improved abstract maximal regularity result for nonautonomous Cauchy problems obtained in [GGS91]. It will be the content of [Saaa].

Moreover, I startet a project together with Y. Giga, K. Inui, A. Mahalov, and S. Matsui on the so-called Ekman boundary layer problem. This concerns the Navier-Stokes equations on the half-space $\mathbb{R}^n_+$ with initial values consisting of the famous Ekman spiral solution added to some function nondecaying in the tangential components. The purpose here is to prove a local existence result for strong solutions of the Navier-Stokes equations.

# References


[DHP01] W. Desch, M. Hieber, and J. Prüss. $L^p$-theory of the Stokes equation in a half-space. *J. Evol. Equ.*, 1:115–142, 2001.

[GGS91] M. Giga, Y. Giga, and H. Sohr. $L^p$ estimate for abstract linear parabolic equations. *Proc. Japan Acad. Ser. A Math. Sci*, 67:197–202, 1991.

[NS03] A. Noll and J. Saal. $H^\infty$-calculus for the Stokes operator on $L_q$-spaces. *Math.Z.*, 244:651–688, 2003.

[Saaa] J. Saal. Maximal regularity for the stokes equations on non-cylindrical space-time domains. In preparation.

[Saab] J. Saal. The Stokes operator with Robin boundary conditions in $L^1_\sigma(\mathbf{R}^n_+)$ and $L^\infty_\sigma(\mathbf{R}^n_+)$. In preparation.

[Saa03] J. Saal. *Robin Boundary Conditions and Bounded $H^\infty$-Calculus for the Stokes Operator*. PhD thesis, TU Darmstadt. Logos Verlag, Berlin, 2003.

[Saa04] J. Saal. Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space. Preprint, Hokkaido University Japan, 2004.

**List of Publications**
of
Jürgen Saal

# References


[1] A. Noll and J. Saal. $H^\infty$-calculus for the Stokes operator on $L_q$-spaces. *Math.Z.*, 244:651–688, 2003.

[2] J. Saal. Maximal regularity for the Stokes equations in non-cylindrical space-time domains. In preparation.

[3] J. Saal. The Stokes operator with Robin boundary conditions in $L^1_\sigma(\mathbf{R}^n_+)$ and $L^\infty_\sigma(\mathbf{R}^n_+)$. In preparation.

[4] J. Saal. The Stokes operator with Robin boundary conditions in $L^\infty_\sigma(\mathbf{R}^n_+)$. Minisymposium on Navier-Stokes Equations and Reaction Diffusion Equations, Equadiff 2003, International Conference on Differential Equations, to appear.

[5] J. Saal. *Robin Boundary Conditions and Bounded $H^\infty$-Calculus for the Stokes Operator.* PhD thesis, TU Darmstadt. Logos Verlag, Berlin, 2003.

[6] J. Saal. Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space. Preprint, Hokkaido University Japan, 2004.

# List of Presentations

of

Jürgen Saal

1. *The Stokes equations in* $\mathbb{R}^n_+$, Workshop on Maximal Regularity and Asymptotic of Elliptic and Parabolic Problems, March (2000), Darmstadt.

2. *Bounded imaginary powers for the Stokes operator in bounded domains*, Workshop on Maximal Regularity and Asymptotic of Elliptic and Parabolic Problems, March (2001), Halle.

3. *$H^\infty$-calculus for the Stokes operator on $L^q$-spaces*, Third European-Maghreb Workshop on Semigroup Theory, Evolution Equations and Applications, March (2002), Marrakesh, Morocco.

4. *$H^\infty$-calculus for the Stokes operator on $L^q$-spaces*, Open Seminar of AG 6, Darmstadt University of Technology, Department of Mathematics, on the occasion of the visit of Prof. Dr. Yoshikazu Giga in Darmstadt, Sepember (2002), Darmstadt.

5. *The Stokes equations with Robin boundary conditions*, Workshop on Modelling and Analysis of Moving Boundaries, December (2002), Wittenberg.

6. *Mixed boundary conditions and functional calculus for the Stokes operator*, Open Seminar of AG 6, Darmstadt University of Technology, Department of Mathematics, April (2003), Darmstadt.

7. *The Stokes operator with Robin Boundary Conditions in $L^\infty_\sigma(\mathbb{R}^n_+)$*, Minisymposium on Navier-Stokes Equations and Reaction Diffusion Equations, Equadiff 2003, International Conference on Differential Equations, July 2003, Hasselt, Belgium.

8. *Robin boundary conditions and bounded $H^\infty$-calculus for the Stokes operator*, Navier-Stokes Seminar, Hokkaido University, Department of Mathematics, October 2, 2003, Sapporo, Japan.

9. *The Stokes operator with Robin boundary conditions in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 \leq q \leq \infty$*, Partial Differential Equations Seminar, Hokkaido University, Department of Mathematics, November 10, 2003, Sapporo, Japan.

10. *$H^\infty$-calculus for the Stokes operator on $L^q$-spaces*, Guest House Minisymposium, Sapporo Guest House, November 15, 2003, Sapporo, Japan.

11. *$H^\infty$-calculus for the Stokes operator on $L^q$-spaces*, Seminar at Waseda University Tokyo, Department of Mathematics, November 18, 2003, Tokyo, Japan.

12. *$H^\infty$-calculus for the Stokes operator on $L^q$-spaces*, Seminar at Saitama University, Department of Mathematics, November 20, 2003, Saitama, Japan.

13. *The Stokes operator with Robin boundary conditions in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 \leq q \leq \infty$*, Seminar at Waseda University Tokyo, Department of Mathematics, November 21, 2003, Tokyo, Japan.

14. Series of lectures about *$H^\infty$-calculus for the Stokes operator on $L^q$-spaces*, Hokkaido University, Department of Mathematics, February 5-6, 2004, Sapporo, Japan.

15. *The Stokes operator with Robin boundary conditions in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 \leq q \leq \infty$*, Seminar at Kanazawa University, Department of Mathematics, February 10, 2004, Kanazawa, Japan.

16. *Maximal regularity for the Stokes system on domains with a moving boundary*, The 5th Northeastern Symposium on Mathematical Analysis, February 23-24, 2004, Sapporo, Japan.